Stabilization of Networked Control Systems with Multirate Sampling[☆]

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Abstract

In this paper, we study the stabilization of networked control systems with multirate sampling. The input channels are modeled in two different ways. First, each of them is modeled as the cascade of a downsampling circuit, an ideal transmission system together with an additive norm bounded uncertainty, and a discrete zero-order hold. Then each input channel is modeled as the cascade of a downsampling circuit, an ideal transmission system together with a feedback norm bounded uncertainty, and a discrete zero-order hold. For each channel model, different downsampling rates are allowed in different input channels. The minimum total channel capacity required for stabilization is investigated. The key idea of our approach is the channel resource allocation, i.e., given the total capacity of the communication network, we do have the freedom to allocate the capacities among different input channels. With this new idea, we successfully show that the multirate networked control system with each channel model can be stabilized by state feedback under an appropriate resource allocation,

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if and only if the total network capacity is greater than the topological entropy of the plant. We also apply the result to multirate quantized control systems. Both the commonly used logarithmic quantizer and the alternative logarithmic quantizer are considered. For each case, a sufficient condition for stabilization is obtained which involves a trade-off between the densities of time quantization and spatial quantization.

Keywords: Networked control system, networked stabilization, multirate sampling, topological entropy, channel resource allocation.

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1. Introduction

Arising from the cross-pollination of control, network and information theories, the networked control systems (NCSs) have attracted great attention nowadays. They are control systems wherein the feedback loop is closed over a communication network. Applications of NCSs have been found in more and more areas. Examples include mobile sensor networks [25], multi-agent systems [21] and automated highway systems [28], *etc.*. In special issues [1, 2] and the survey papers [12, 23, 13], much information of the current status of NCSs research has been presented.

In the NCSs, different kinds of information constraints and uncertainties appear due to the imperfect communication networks, such as quantization [10, 11], packet drop [9, 30, 32], limited data rate [19, 22] and delay [24, 34], *etc.*. Numerous results have been reported in the literature addressing the stabilization of NCSs under these constraints and uncertainties. For discrete-time single-input NCSs, in [11], logarithmic quantization of the control inputs is considered as a sector uncertainty. It is shown that the largest uncertainty bound which renders stabilization possible is given in terms of the Mahler measure of the system, i.e., the absolute product of the unstable poles. The NCS with multiplicative stochastic input channel is studied in [9] which states that the NCS can be mean-square stabilized by state feedback, if and only if the mean-square channel capacity exceeds the topological entropy of the plant which is the logarithm of the Mahler measure. The networked stabilization over additive white Gaussian noise (AWGN) channel is studied in [4], where the minimum channel capacity rendering stabilization possible for the single-input case is given again in terms of the topological entropy of the plant.

For discrete-time multi-input NCSs, the work in [26] models each input channel in three different ways, i.e., the signal-to-error ratio (SER) model, the received signal-to-error ratio (R-SER) model and the AWGN channel model. The main contribution there is in the introduction of the channel resource allocation to solve the networked stabilization problem. It is assumed that the information constraints in the input channels are determined by the total network recourse available to the channels which can be allocated by the controller designer. This additional design freedom gives rise to channel/controller co-design, under which a uniform analytic solution is obtained for the minimum total channel capacity required for stabilization with each channel model given again in terms of the topological entropy of the plant. The multi-input NCSs with multiplicative stochastic input channels are studied in [32] which generalizes the stabilization condition for the single input case [9] to the multi-input case by applying the channel resource allocation.

Researchers have also devoted much effort to the continuous-time networked stabilization. Reference [33] studies stabilization of a continuous-time LTI system over multiplicative stochastic input channels with channel resource allocation leading to the minimum total capacity required for stabilization also given by the topological entropy of the plant. A distributed control system is investigated in [7] where a central controller communicates sequentially with the subsystems through one shared communication network under some periodic communication pattern. Both the communication pattern and the control law are to be designed, leading to channel/controller co-design for periodic multirate linear systems. Another work involving multirate operations in NCSs can be seen in [15] where a subband coding scheme is proposed to efficiently use the available bit rates and to account for message losses. The trade-off between the required densities of time quantization and spatial quantization for stabilization of NCSs has also been studied in the literature, which is closely related to our work in this paper. For single-input case, the situation of uniform sampling and infinite-level logarithmic spatial quantization is considered in [10] and a trade-off between the densities is obtained in terms of the Mahler measure. In the case when a finite-level spatial quantizer is used, the trade-off is studied in [16, 17]. There it is concluded that the minimum data rate for stabilization could only be achieved by binary control. Unfortunately, so far, no efficient result has been reported on the trade-off for the multi-input case.

Inspired by the existing results discussed above, we in this paper study stabilization of continuous-time NCSs with multirate sampling. Partial results of this study have been reported in [5]. In this work, two different channel models are adopted. The first one is the cascade of a downsampling circuit, an ideal transmission system together with an additive norm bounded uncertainty, and a discrete zero-order hold. Although this model is motivated from the logarithmic quantizer studied in [10, 11], it also has the capability to address other network features. The second model is the cascade of a downsampling circuit, an ideal transmission system together with a feedback norm bounded uncertainty, and a discrete zero-order hold. This model is motivated from an alternative logarithmic quantizer. Each channel model consists of three components with the second component inherited from the SER model or the R-SER model proposed in [26]. Different from [26] that focuses on discrete-time NCSs, in this paper, we start with a continuous-time multi-input system. The additional downsampling and hold components in the input channels enable multirate sampling leading to a multirate NCS. The main novelty of this work is to investigate the minimum total channel capacity required for stabilization with channel resource allocation, i.e., the capacities can be allocated among different input channels. Lifting technique is employed to transform the multirate system to an equivalent LTI system. We show that for each channel model, the multirate NCS could be stabilized by state feedback under an appropriate resource allocation, if and only if the total network capacity is greater than the topological entropy of the plant. We further apply this result to multirate quantized control systems. Both the commonly used logarithmic quantizer and the alternative logarithmic quantizer are considered. For each case, a sufficient condition for stabilization is obtained which shows a trade-off between the densities of time quantization and spatial quantization.

Note that the idea of channel resource allocation was first proposed in the conference paper [14] to study the stabilization of multi-input NCSs and then extended in [26]. Following this idea, several other works have been carried out, e.g., [32, 33, 35, 5, 6].

The remainder of this paper is organized as follows. The multirate NCS is formulated in section 2. Some preliminary knowledge on multirate systems and lifting technique are presented in section 3. The main result on minimum capacity required for stabilization is stated and proved in section 4. Section 5 applies the result to the trade-off between the densities of time quantization and spatial quantization. Section 6 gives an illustrative example. Finally, some conclusion remarks follow in section 7. The notations in this paper is more or less standard and will be made clear as we proceed.

2. Problem Formulation

The setup of a multirate NCS studied in this paper is shown in Figure 1. We use solid lines for continuous-time signals and dotted lines for discrete-time signals. The plant is a continuous-time LTI system with state space realization $[A|B]$:

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$. The sampled states $x_d(k) = x(kT)$ are available for feedback with sampling interval T. Assume that all hold and sampling circuits are synchronized at time 0. The control signal $v(k)$ generated by a static state feedback gain F is transmitted through a multirate communication network before reaching the plant. In many practical applications, the actuators are located separately from each other and from the controller. To fit this case, a parallel transmission strategy is adopted, i.e., each element $v_i(k)$ of the control signal is separately sent through an independent communication channel. The received control signal is finally converted to a continuous-time signal by a zero-order hold with period T.

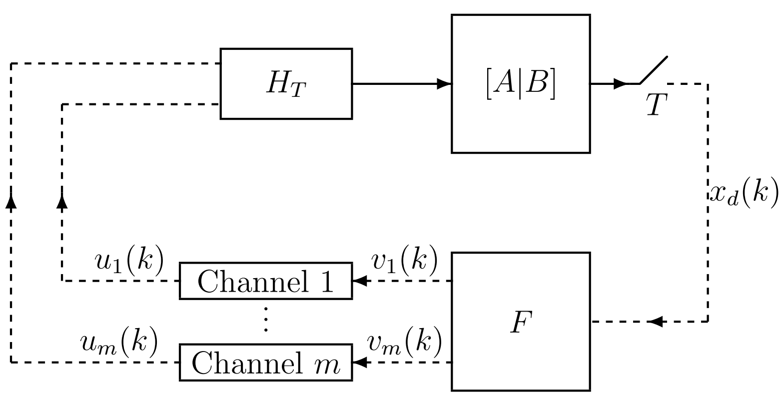

Figure 1: A multirate NCS.

In this paper, the communication channels are modeled in two different ways. The first model, depicted in Figure 2, is the cascade of a downsampling circuit, an ideal transmission system with a unity transfer function together with an additive norm bounded uncertainty, and a discrete zero-order hold.

The uncertainty $\mathbf{\Delta}_i$ can be a nonlinear, time-varying and dynamic system. The only assumption is that $\mathbf{\Delta}_i(0) = 0$ is the unique equilibrium point and its $\mathcal{H}_\infty$ norm

$$\|\mathbf{\Delta}_i\|_\infty = \sup_{\tilde{v}_i \in \ell_2} \frac{\|e_i\|_2}{\|\tilde{v}_i\|_2} \leq \delta_i$$

for some δ_i. As remarked in [26], the channel introduces a multiplicative uncertainty to the plant. The inverse uncertainty bound δ_i^{-1} can be considered as the worst case SER. One of the novelties of this paper is that we allow different downsampling rates K_i in different input channels. Without loss of generality, assume that $K_1, K_2, \ldots, K_m$ are relative prime integers. The advantage of multirate sampling stands out not only in theoretical studies but also in practical applications. For example, multirate downsampling is adopted in [15] to efficiently use the limited data rates in NCSs. From the practical perspective, in complex, multivariable control systems, sampling all physical signals uniformly at one single rate is often unrealistic, then one is forced to use multirate sampling. Also, multirate sampling can often reduce the required storage space or computational complexity for signal processing.

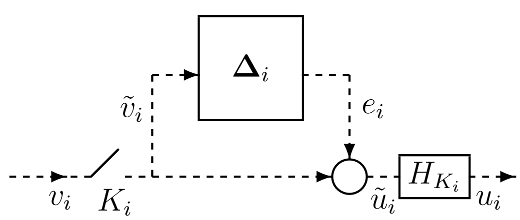

Figure 2: An SER channel model.

We define the capacity of channel i to be

$$\mathfrak{C}_i = \frac{1}{T_i} \ln \delta_i^{-1}, i = 1, 2, \ldots, m,$$

where $T_i = K_i T$. This capacity depends linearly on the sampling frequency $\frac{1}{T_i}$ and the logarithm of the inverse uncertainty bound δ_i^{-1}. It measures how much information per time unit can be transmitted through the ith channel. To measure the amount of information transmitted through the whole network per time unit, we define the total network capacity by summing up all the capacities $\mathfrak{C}_i$, i.e., $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

This channel model is motivated from the use of the logarithmic quantizer given by the following nonlinear mapping [10]:

$$\tilde{u}_i = Q_{\delta_i}(\tilde{v}_i) := \begin{cases} \rho_i^l \xi_i, & \text{if } \frac{\rho_i^l \xi_i}{1+\delta_i} < \tilde{v}_i \leq \frac{\rho_i^l \xi_i}{1-\delta_i}, \\ 0, & \text{if } \tilde{v}_i = 0, \\ -Q_{\delta_i}(-\tilde{v}_i), & \text{if } \tilde{v}_i < 0, \end{cases} \tag{1}$$

where $\xi_i > 0$, $0 < \rho_i < 1$, $\delta_i = \frac{1-\rho_i}{1+\rho_i}$, and $l = 0, \pm 1, \pm 2, \ldots$. However, it can capture not only the logarithmic quantizer but also other unknown transmission and actuation errors as well.

We are interested in finding the minimum capacities $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ so as to make stabilization of the multirate NCS possible. Intuitively, if the channel capacities are too small, i.e., sampling rates are too slow or the uncertainty bounds are too large, then little information of the control signals can be transmitted and the multirate NCS can hardly be stabilized. Only when enough information is transmitted per time unit can stabilization become possible.

Our second channel model, depicted in Figure 3, is the cascade of a downsampling circuit, an ideal transmission system with a unity transfer function together with a feedback norm bounded uncertainty, and a discrete zero-order hold. Again, different downsampling rates K_i are allowed in different input channels and the uncertainty $\mathbf{\Delta}_i$ can be a nonlinear, time-varying and dynamic system. The only assumption is that $\mathbf{\Delta}_i(0) = 0$ is the unique equilibrium point and its $\mathcal{H}_\infty$ norm

$$\|\mathbf{\Delta}_i\|_\infty = \sup_{\tilde{u}_i \in \ell_2} \frac{\|e_i\|_2}{\|\tilde{u}_i\|_2} \leq \delta_i$$

for some δ_i. Different from the first model, the channel now introduces a relative uncertainty instead of a multiplicative uncertainty to the plant input. The inverse uncertainty bound δ_i^{-1} can be considered as the worst case R-SER [26] instead of the worst case SER. This implies that the two channel models have different physical meanings even if they have the same norm bound. For the second channel model, we define the capacity of channel i as

$$\mathfrak{C}_i = \frac{1}{T_i} \ln \delta_i^{-1}, i = 1, 2, \ldots, m,$$

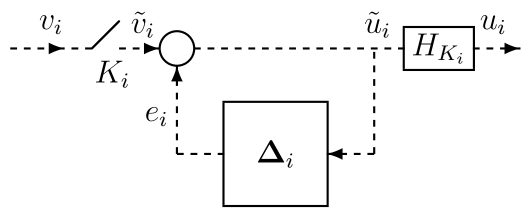

Figure 3: An R-SER channel model.

where $T_i = K_i T$. Summing up all the capacities $\mathfrak{C}_i$ gives rise to the total network capacity $\mathfrak{C} = \sum_{i=1}^{m} \mathfrak{C}_i$.

The second channel model is motivated from the use of an alternative logarithmic quantizer advocated in [26] which is given by the following nonlinear mapping:

$$\tilde{u}_i = \tilde{Q}_{\delta_i}(\tilde{v}_i) := \begin{cases} \rho_i^l \xi_i, & \text{if } \rho_i^l \xi_i (1-\delta_i) < \tilde{v}_i \le \rho_i^l \xi_i (1+\delta_i), \\ 0, & \text{if } \tilde{v}_i = 0, \\ -\tilde{Q}_{\delta_i}(-\tilde{v}_i), & \text{if } \tilde{v}_i < 0, \end{cases} \tag{2}$$

where $\xi_i > 0$, $0 < \rho_i < 1$, $\delta_i = \frac{1-\rho_i}{1+\rho_i}$, and $l = 0, \pm 1, \pm 2, \ldots$. One can refer to [26] for a comparison of this alternative logarithmic quantizer and the commonly used logarithmic quantizer.

Again, we are interested in finding the minimum capacities $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ so as to make stabilization of the multirate NCS possible.

Before proceeding, we define the topological entropy of a continuous-time LTI system. Recall that the Mahler measure [18] of a matrix $A \in \mathbb{R}^{n\times n}$ is

$$M(A) = \prod_{i=1}^{n} \max\{1, |\lambda_i|\},$$

and the topological entropy [3] of A is given by

$$h(A) = \ln M(A) = \sum_{|\lambda_i|>1} \ln |\lambda_i|,$$

where λ_i are the eigenvalues of A. Here, we take the natural logarithm to be consistent with the channel capacity notion defined before. In fact, the base of the logarithm does not affect our main result except for multiplication by a constant. Based on the topological entropy of a linear map, we define the topological entropy of a continuous-time system $\dot{x}(t) = Ax(t)$ as

$$H_c(A) = h(e^A) = \sum_{\Re(\lambda_i)>0} \lambda_i,$$

where λ_i are the eigenvalues of A.

3. Preliminary - multirate systems and lifting

Consider the multirate NCS in Figure 1. Recall that the downsampling rates K_i in different input channels are relative prime. Let

$$N = \text{LCM}\{K_1, K_2, \ldots, K_m\},$$

where LCM means the least common multiple. We can interpret N as the least common period for the downsampling-hold scheme in different channels.

Lifting [8] is a common and efficient technique to deal with multirate sampling by converting the multirate system into an equivalent LTI system with extended input and output dimensions. Specifically, let ℓ be the space of sequences, perhaps vector valued, defined on the time set $\{0, 1, 2, \ldots\}$. The lifting operator over ℓ is given by

$$L_p : \{u(0), u(1), u(2), \ldots\} \mapsto \left\{ \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(p-1) \end{bmatrix}, \begin{bmatrix} u(p) \\ u(p+1) \\ \vdots \\ u(2p-1) \end{bmatrix}, \ldots \right\}.$$

Note that the lifting operator is invertible and norm preserving [8].

Discretizing the plant $[A|B]$ with period T yields the discretized system:

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), x_d(0) = x_0,$$

where $A_d = e^{AT}, B_d = \int_0^T e^{A(T-\tau)} B d\tau$. Denote B_{d_j} as the jth column of B_d. Since N is the shortest period for which the multirate downsampling-hold scheme in the input channels repeats itself, we lift the discretized plant to time period N leading to the following equivalent system with state $x_e(k) = x_d(kN)$:

$$x_e(k+1) = A_e x_e(k) + B_e u_e(k), x_e(0) = x_0,$$

where

$$\begin{aligned} A_e =& A_d^N, \quad B_e = [B_{e_1} \quad B_{e_2} \quad \dots \quad B_{e_m}], \\ B_{e_j} =& \left[A_d^{N-1} B_{d_j} \quad A_d^{N-2} B_{d_j} \quad \dots \quad B_{d_j}\right], \end{aligned}$$

$$u_e(k) = \begin{bmatrix} u_{d_1}(kN) \\ u_{d_1}(kN+1) \\ \vdots \\ u_{d_1}(kN+N-1) \\ u_{d_2}(kN) \\ u_{d_2}(kN+1) \\ \vdots \\ u_{d_2}(kN+N-1) \\ \vdots \\ u_{d_m}(kN+N-1) \end{bmatrix}.$$

Examining the detailed structure of $u_e(k)$ reveals that it is obtained by lifting the inputs of each channel first and then grouping them all together. Clarifying this would make it easier to understand the later design of the state feedback gain F.

The control signal is generated by the feedback law $v(k) = Fx_d(k)$. Let F_i be the ith row of F. To see the behavior of the controller in the time period N, we apply the lifting technique to get

$$v_e(k) = F_e x_e(k) = \begin{bmatrix} F_1 \\ F_1 A_d \\ \vdots \\ F_1 A_d^{N-1} \\ F_2 \\ F_2 A_d \\ \vdots \\ F_2 A_d^{N-1} \\ \vdots \\ F_m A_d^{N-1} \end{bmatrix} x_e(k). \tag{3}$$

Clearly, $v_e(k)$ is the lifted controller output. We will come back to the structure of F_e as shown in (3) when we design the feedback controller in the next section.

As introduced before, different components of the control signal are transmitted through independent communication channels with different downsampling rates K_i. Denote $N_i = \frac{N}{K_i}$. For the SER channel model in Figure 2, applying the lifting technique to the transmission process yields

$$u_e = \mathscr{H}(\boldsymbol{I} + \underline{\boldsymbol{\Delta}})\mathscr{S}v_e,$$

where $\boldsymbol{I}$ is the identity system, $\underline{\boldsymbol{\Delta}} = \text{diag}\{\underline{\boldsymbol{\Delta}_1}, \underline{\boldsymbol{\Delta}_2}, \dots, \underline{\boldsymbol{\Delta}_m}\}$ is the lifted uncertainty with $\underline{\boldsymbol{\Delta}_i} = L_{N_i}\Delta_i L_{N_i}^{-1}$, $\mathscr{S} = \text{diag}\{\mathscr{S}_1, \mathscr{S}_2, \dots, \mathscr{S}_m\}$ describes the downsampling scheme of the input channels with $\mathscr{S}_i$ having dimension $N_i \times N$, and $\mathscr{H} = \text{diag}\{\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_m\}$ describes the hold scheme with $\mathscr{H}_i$ having dimension $N \times N_i$. Let $\mathscr{S}_i^{jk}$ and $\mathscr{H}_i^{jk}$ be the (j,k)th element of $\mathscr{S}_i$ and $\mathscr{H}_i$ respectively, then

$$\mathscr{S}_i^{jk} = \begin{cases} 1 & \text{when } k = (j-1)K_i + 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{H}_i^{jk} = \begin{cases} 1 & \text{when } (k-1)K_i + 1 \leq j \leq kK_i, \\ 0 & \text{otherwise.} \end{cases}$$

Since lifting is norm preserving, we have $\|\underline{\boldsymbol{\Delta}_i}\|_\infty \leq \delta_i$.

Similarly, for the R-SER channel model in Figure 3, applying the lifting technique to the transmission process yields

$$u_e = \mathscr{H}(\boldsymbol{I} - \underline{\boldsymbol{\Delta}})^{-1}\mathscr{S}v_e,$$

where $\underline{\boldsymbol{\Delta}}$, $\mathscr{S}$ and $\mathscr{H}$ have the same expression as in the SER model case.

So far, we have obtained quite much knowledge on the structure of the multirate NCS. The lifted closed-loop system would follow directly from the equivalent LTI systems associated with the plant, controller and network.

Throughout the rest of this paper, the following assumption is made: NT is nonpathological with respect to A, i.e., $\lambda_i - \lambda_j \neq \frac{2k\pi\sqrt{-1}}{NT}, k = 1, 2, \dots,$ for any two eigenvalues λ_i and λ_j of A [8]. This mild assumption on the nonpathological sampling period is to make sure that the lifted system $[A_e|B_e]$ does not lose the stabilizability [27]. In view of [20], the closed-loop multirate NCS is stable if and only if the lifted closed-loop system is stable. Therefore, our problem becomes to find the stabilizing conditions for the lifted system, which will be solved in the next section.

Before moving on, it is worth briefly reviewing another useful technique called Wonham decomposition. It was originally put forward in [31] to solve the multi-input pole placement problem. Given a continuous-time stabilizable multi-input system $[A|B]$, we can carry out the controllable-uncontrollable decomposition with respect to the first column of B by a similarity transformation such that $[A|B]$ is equivalent to

$$\left[\begin{bmatrix} A_1 & * \\ 0 & \tilde{A}_2 \end{bmatrix} \middle| \begin{bmatrix} b_1 & * \\ 0 & \tilde{B}_2 \end{bmatrix}\right].$$

Then we proceed to do the controllable-uncontrollable decomposition to the system $[\tilde{A}_2|\tilde{B}_2]$ with respect to the first column of $\tilde{B}_2$. Continuing this process yields the following Wonham decomposition

$$\left[\begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}\right], \tag{4}$$

that is equivalent to $[A|B]$, where each subsystem $[A_i|b_i]$ is stabilizable.

4. Minimum capacity for stabilization of multirate NCS

This section is to serve the main purpose of this paper, i.e., finding the minimum channel capacity required for stabilization of the multirate NCS with each channel model. As we mentioned before, this is equivalent to find the minimum channel capacity so that a state feedback gain F can be designed to make the lifted closed-loop system stable. Due to the existence of more than one uncertainties in the loop, the stabilization will involve a μ-synthesis problem which is very difficult to solve. However, this difficulty can be mitigated by the idea of channel resource allocation, as elaborated in the rest of this section.

4.1. SER model

By the $\mathcal{H}_\infty$ robust control theory, stabilization of the multirate NCS with the SER channel model involves $\mathcal{H}_\infty$ optimization of the complementary sensitivity function $T(z)$ of the lifted feedback system, where

$$T(z) = \mathscr{S} F_e(zI - A_e - B_e \mathscr{H} \mathscr{S} F_e)^{-1} B_e \mathscr{H}.$$

If the uncertainty bounds $\delta_1, \delta_2, \ldots, \delta_m$ and the state feedback gain F_e are given, the uncertain system is stabilized for all possible uncertainties satisfying the bounds if and only if [29]

$$\inf_{D \in \mathcal{D}} \left\| D^{-1} T(z) D \Psi \right\|_\infty < 1, \tag{5}$$

where $\mathcal{D}$ is the set of all diagonal matrices with the structure

$$\operatorname{diag}\{d_1 I_{N_1}, d_2 I_{N_2}, \ldots, d_m I_{N_m}\} \tag{6}$$

and

$$\Psi = \operatorname{diag}\{\delta_1 I_{N_1}, \delta_2 I_{N_2}, \ldots, \delta_m I_{N_m}\}. \tag{7}$$

Note that if the uncertainties are specifically caused by logarithmic quantizers, since the quantizers are static without any dynamics, the inequality (5) may not be necessary for stability of the closed-loop system. We will come across this situation when we study the multirate quantized control systems in the next section.

The minimization problem in (5) can be converted to a convex problem and is hence manageable. However, the design problem, i.e., to find a stabilizing F_e such that (5) holds, is very difficult. We can formulate the design problem as the following minimization problem:

$$\inf_{F_e \text{ stabilizing}} \left[\inf_{D \in \mathcal{D}} \left\| D^{-1} T(z) D \Psi \right\|_\infty \right], \tag{8}$$

where the infimum is taken over the set of all stabilizing state feedback gain F_e in the sense that $A_e + B_e \mathscr{H} \mathscr{S} F_e$ is stable. Unfortunately, the objective function in (8) cannot be converted to a jointly convex problem. In fact, a deeper look reveals the root of the difficulty: when the uncertainty bounds are specified a priori, a μ-synthesis problem emerges which is notoriously hard.

The channel resource allocation provides a significant insight to handle this difficulty. In the NCSs, quite often the capacity of a channel is closely related to the resource available to it. If we allocate more resource to one channel, e.g., use better and more expensive hardware or allocate more communication bandwidth, then we are able to increase its capacity. Considering this situation, it is natural and reasonable to consider the total channel capacity required for stabilization assuming that the capacities can be allocated among different channels. In other words, instead of specifying a priori the capacity of each channel, an overall constraint on the total capacity is given and the controller designer can allocate the channel capacities in an optimal way to facilitate the design of the controller. For our current problem, allocating the channel capacity involves two aspects. One is allocating the downsampling rates and the other is allocating the uncertainty bounds. By looking into the structure of Ψ, we find these two aspects are simultaneously contained in Ψ. Therefore, the overall constraint on the total capacity can be given in terms of $\delta = \det \Psi = \Pi_{i=1}^{m} \delta_i^{N_i}$. Applying the channel resource allocation yields a further nested minimization problem:

$$\inf_{\det \Psi = \delta} \left\{ \inf_{F_e \text{ stabilizing}} \left[\inf_{D \in \mathcal{D}} \left\| D^{-1} T(z) D \Psi \right\|_\infty \right] \right\}.$$

At first sight, this problem looks even harder than problem (8), however, surprisingly, it can be analytically solved, as shown in the following theorem.

Theorem 1. *The multirate NCS with SER channel model is stabilizable by state feedback under an appropriate channel resource allocation, if and only if* $\mathfrak{C} > H_c(A)$.

Proof. For brevity, assume that all the eigenvalues of A lie on the open right half complex plane. This assumption can be removed following the same argument as in [26]. Under this assumption, all the eigenvalues of A_e lie outside the unit circle. By [27], $[A_e|B_e]$ is stabilizable if $[A|B]$ is stabilizable when NT is nonpathological with respect to A as assumed.

To show the necessity part, assume that there exists a stabilizing state feedback gain F_e and a $D \in \mathcal{D}$ such that

$$\|D^{-1} T(z) D \Psi\|_\infty < 1, \tag{9}$$

then it is shown in [26] that $\delta^{-1} > M(A_e)$. Since $\delta^{-1} = \Pi_{i=1}^{m} (\delta_i^{-1})^{N_i}$, $M(A_e) = M(A_d^N) = e^{NT \sum \lambda_i}$, after some calculations, we have $\delta^{-1} > M(A_e)$ if and only if $\mathfrak{C} > H_c(A)$.

To show the sufficiency part, for any given $\mathfrak{C} > H_c(A)$, we find a $D \in \mathcal{D}$, a stabilizing state feedback gain F_e and a factorization $\delta = \Pi_{i=1}^{m} \delta_i^{N_i}$ such that (9) holds. Without loss of generality, we assume that $[A|B]$ has the Wonham decomposition given by (4), where each subsystem $[A_i|b_i]$ is stabilizable with state dimension n_i. Then the lifted system $[A_e|B_e]$ has the following decomposition:

$$\left[\begin{bmatrix} A_{e_1} & * & \cdots & * \\ 0 & A_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_{e_m} \end{bmatrix} \middle| \begin{bmatrix} b_{e_1} & * & \cdots & * \\ 0 & b_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_{e_m} \end{bmatrix}\right]. \tag{10}$$

Since $[A_i|b_i]$ is stabilizable, it follows that $[A_{e_i}|b_{e_i}]$ is stabilizable for all $i = 1, 2, \ldots, m$.

Choose

$$D = \mathrm{diag}\{I_{N_1}, \epsilon I_{N_2}, \ldots, \epsilon^{m-1} I_{N_m}\} \tag{11}$$

with a small positive real number ϵ. Also define

$$P = \mathrm{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{m-1} I_{n_m}\}. \tag{12}$$

Let $\tilde{F}_e = D^{-1}\mathscr{S} F_e, \tilde{B}_e = B_e \mathscr{H} D$, then

$$\begin{aligned} D^{-1}T(z)D\Psi &= \tilde{F}_e(zI - A_e - \tilde{B}_e\tilde{F}_e)^{-1}\tilde{B}_e\Psi \\ &= \tilde{F}_e P(zI - P^{-1}A_eP - P^{-1}\tilde{B}_e\tilde{F}_eP)^{-1}P^{-1}\tilde{B}_e\Psi, \end{aligned}$$

where

(13)

$$P^{-1}A_eP = \begin{bmatrix} A_{e_1} & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & A_{e_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & A_{e_m} \end{bmatrix}, P^{-1}\tilde{B}_e = \begin{bmatrix} b_{e_1}\mathscr{H}_1 & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & b_{e_2}\mathscr{H}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & b_{e_m}\mathscr{H}_m \end{bmatrix},$$

and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$.

Since $\mathfrak{C} > H_c(A)$, i.e., $\delta < M(A_e)^{-1}$, we can always possibly choose δ_i such that $\delta_i^{N_i} < M(A_{e_i})^{-1}, i = 1, 2, \ldots, m$ and $\delta = \Pi_{i=1}^m \delta_i^{N_i}$. This in fact realizes the allocation of the individual channel capacity $\mathfrak{C}_i$ such that $\mathfrak{C}_i > H_c(A_i)$ and $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$. With this allocation of capacity, we consider each single-input NCS corresponding to $[A_i|b_i]$. Discretizing $[A_i|b_i]$ with time period K_iT yields a discretized system $[A_{s_i}|b_{s_i}]$:

$$A_{s_i} = A_{d_i}^{K_i}, \quad b_{s_i} = \sum_{q=1}^{K_i} A_{d_i}^{K_i - q} b_{d_i}, \tag{14}$$

where $A_{d_i} = e^{A_i T}, b_{d_i} = \int_0^T e^{A_i(T-\tau)} b_i d\tau$. Since $\delta_i^{N_i} < M(A_{e_i})^{-1}$, it follows directly that $\delta_i < M(A_{s_i})^{-1}$. According to Lemma 2 in [26], a state feedback gain f_i could be designed such that $[A_{s_i}|b_{s_i}]$ is stabilized for all uncertainties satisfying the norm bound δ_i and the following inequality holds:

$$\|f_i(zI - A_{s_i} - b_{s_i} f_i)^{-1} b_{s_i}\|_\infty \delta_i < 1.$$

Applying the lifting technique in accordance with time period N yields the lifted feedback gain

$$f_{e_i} = \begin{bmatrix} f_i \\ f_i A_{d_i} \\ \vdots \\ f_i A_{d_i}^{N-1} \end{bmatrix} \tag{15}$$

and the lifted complementary sensitivity function

$$T_i(z) = \mathscr{S}_i f_{e_i} (zI - A_{e_i} - b_{e_i} \mathscr{H}_i \mathscr{S}_i f_{e_i})^{-1} b_{e_i} \mathscr{H}_i.$$

Since the lifting operator preserves norms, we have $\|T_i(z)\delta_i\|_\infty < 1$.

Let $F = \text{diag}\{f_1, f_2, \ldots, f_m\}$. In view of the structure of F_e in (3), we get $\tilde{F}_e P = D^{-1} \mathscr{S} F_e P = \text{diag}\{\mathscr{S}_1 f_{e_1}, \mathscr{S}_2 f_{e_2}, \ldots, \mathscr{S}_m f_{e_m}\} + o(\epsilon)$. It can now be verified that

$$D^{-1} T(z) D \Psi = \text{diag}\{T_1(z)\delta_1, T_2(z)\delta_2, \ldots, T_m(z)\delta_m\} + o(\epsilon; z).$$

Since $\|T_i(z)\delta_i\|_\infty < 1$ and $o(\epsilon; z) \to 0$ as $\epsilon \to 0$ for each $|z| \geq 1$, it follows that $\|D^{-1} T(z) D \Psi\|_\infty < 1$ for sufficiently small ϵ which completes the proof.

The overall process of channel resource allocation and controller design as shown in the above proof constitutes channel/controller co-design. The controller designer should also participate in the channel design rather than passively take the channels given by the system designer. With this co-design, the difficulty caused by the μ-synthesis problem can be mitigated and the minimum total channel capacity required for stabilization is obtained. Specifically, for a given total capacity $\mathfrak{C} > H_c(A)$, a feasible allocation of $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ so that $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$ is to make $\mathfrak{C}_i > H_c(A_i)$. To be more precise, the channel/controller co-design is carried out in the following way: choose $\mathfrak{C}_1$ and design f_1 so that the first input is used to stabilize all unstable modes controllable from the first input; choose $\mathfrak{C}_2$ and design f_2 so that the second input is used to stabilize the additional unstable modes controllable from the second input excluding the ones that are already stabilized by the first input; ...; finally, choose $\mathfrak{C}_m$ and design f_m to stabilize the remaining unstable modes that are not stabilized by the other inputs.

Recall that the allocation of capacities involves both the allocation of downsampling rates K_i and the allocation of uncertainty bounds δ_i. In fact, there exists a tradeoff between the choice of K_i and δ_i. The system with faster sampling rates can tolerant more uncertainty in terms of stabilization. Theoretically, K_i and δ_i can be arbitrarily chosen as long as $\mathfrak{C}_i > H_c(A_i)$. However, from practical perspective, sampling rates cannot be arbitrarily slow due to the limitations on the uncertainty bounds.

4.2. R-SER model

Different from the SER model case, stabilization of the multirate NCS with the R-SER channel model involves $\mathcal{H}_\infty$ optimization of the sensitivity function $S(z)$ of the lifted feedback system, where

$$S(z) = I + \mathscr{S} F_e (zI - A_e - B_e \mathscr{H} \mathscr{S} F_e)^{-1} B_e \mathscr{H}.$$

Precisely, for given uncertainty bounds $\delta_1, \delta_2, \ldots, \delta_m$ and a stabilizing state feedback gain F_e, the uncertain system is stabilized for all possible uncertainties satisfying the bounds if and only if

$$\inf_{D \in \mathcal{D}} \left\| D^{-1} S(z) D \Psi \right\|_\infty < 1, \tag{16}$$

where $\mathcal{D}$ is the set of all diagonal matrices with the structure in (6) and Ψ is given by (7).

Similar to the SER model case, due to the existence of multiple uncertainties in the loop, a μ-synthesis problem arises which is very difficult to solve. Again, the idea of channel resource allocation can mitigate this difficulty. The overall constraint of total channel capacity is given in terms of $\delta = \det \Psi = \Pi_{i=1}^m \delta_i^{N_i}$. Applying the channel resource allocation yields the following minimization problem:

$$\inf_{\det \Psi = \delta} \left\{ \inf_{F_e \text{ stabilizing}} \left[\inf_{D \in \mathcal{D}} \left\| D^{-1} S(z) D \Psi \right\|_\infty \right] \right\}.$$

This problem, again, admits a very nice analytic solution.

Theorem 2. *The multirate NCS with R-SER channel model is stabilizable by state feedback under an appropriate channel resource allocation, if and only if* $\mathfrak{C} > H_c(A)$.

Proof. As in the proof of Theorem 1, assume that all the eigenvalues of A lie on the open right half complex plane.

To show the necessity part, assume that there exists a stabilizing state feedback gain F_e and a $D \in \mathcal{D}$ such that

$$\|D^{-1}S(z)D\Psi\|_\infty < 1, \tag{17}$$

then it is shown in [26] that $\delta^{-1} > M(A_e)$ which is equivalent to $\mathfrak{C} > H_c(A)$.

To show the sufficiency part, for any given $\mathfrak{C} > H_c(A)$, we find a $D \in \mathcal{D}$, a stabilizing state feedback gain F_e and a factorization $\delta = \Pi_{i=1}^m \delta_i^{N_i}$ such that (17) holds. As in the proof of Theorem 1, we assume that $[A|B]$ has the Wonham decomposition given by (4). Choose D as in (11) and define P as in (12), then

$$\begin{aligned} D^{-1}S(z)D\Psi &= \left(I + \tilde{F}_e(zI - A_e - \tilde{B}_e\tilde{F}_e)^{-1}\tilde{B}_e\right)\Psi \\ &= \left(I + \tilde{F}_e P(zI - P^{-1}A_eP - P^{-1}\tilde{B}_e\tilde{F}_eP)^{-1}P^{-1}\tilde{B}_e\right)\Psi, \end{aligned}$$

where $\tilde{F}_e = D^{-1}\mathscr{S}F_e, \tilde{B}_e = B_e\mathscr{H}D$ and $P^{-1}A_eP, P^{-1}\tilde{B}_e$ are given by (13).

Since $\mathfrak{C} > H_c(A)$, i.e., $\delta < M(A_e)^{-1}$, we can always possibly choose δ_i such that $\delta_i^{N_i} < M(A_{e_i})^{-1}, i = 1, 2, \ldots, m$ and $\delta = \Pi_{i=1}^m \delta_i^{N_i}$. Since $\delta_i^{N_i} < M(A_{e_i})^{-1}$, it follows directly that $\delta_i < M(A_{s_i})^{-1}$. According to Lemma 2 in [26], a state feedback gain f_i could be designed such that $[A_{s_i}|b_{s_i}]$ as in (14) is stabilized for all uncertainties satisfying the norm bound δ_i and the following inequality holds:

$$\|I + f_{s_i}(zI - A_{s_i} - b_{s_i}f_{s_i})^{-1}b_{s_i}\|_\infty \delta_i < 1.$$

Applying the lifting technique in accordance with time period N yields the lifted sensitivity function

$$S_i(z) = I + \mathscr{S}_i f_{e_i}(zI - A_{e_i} - b_{e_i}\mathscr{H}_i\mathscr{S}_i f_{e_i})^{-1}b_{e_i}\mathscr{H}_i,$$

where f_{e_i} is the lifted feedback gain as in (15). Since the lifting operator preserves norms, we have $\|S_i(z)\delta_i\|_\infty < 1$.

Let $F = \mathrm{diag}\{f_1, f_2, \ldots, f_m\}$. In view of the structure of F_e in (3), we get $\tilde{F}_e P = D^{-1}\mathscr{S} F_e P = \mathrm{diag}\{\mathscr{S}_1 f_{e_1}, \mathscr{S}_2 f_{e_2}, \ldots, \mathscr{S}_m f_{e_m}\} + o(\epsilon)$. It can now be verified that

$$D^{-1}S(z)D\Psi = \mathrm{diag}\{S_1(z)\delta_1, S_2(z)\delta_2, \ldots, S_m(z)\delta_m\} + o(\epsilon; z).$$

Since $\|S_i(z)\delta_i\|_\infty < 1$ and $o(\epsilon; z) \to 0$ as $\epsilon \to 0$ for each $|z| \geq 1$, it follows that $\|D^{-1}S(z)D\Psi\|_\infty < 1$ for sufficiently small ϵ which completes the proof.

Note that the remarks following Theorem 1 on the implementation of the channel/controller co-design and the tradeoff between the downsampling rates and the uncertainty bounds also apply here.

It is worth stressing that although the minimum total channel capacity required for stabilization is the same as that in the SER model case, the optimal feedback gain minimizing $\|S(z)\|_\infty$ is different from that minimizing $\|T(z)\|_\infty$. Moreover, it has been shown in [26] that optimizing $\|S(z)\|_\infty$ is preferred to optimizing $\|T(z)\|_\infty$. The reason is that optimizing $\|S(z)\|_\infty$ shares a common optimal feedback gain with the optimization of $\|T(z)\|_2$ and $\|S(z)\|_2$ involved in many other NCSs, e.g., the ones with fading input channels or those perturbed by additive white noises. In this sense, the study with the R-SER channel model provides future potential to investigate NCSs involving mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. In contrast, optimizing $\|T(z)\|_\infty$ is conflicting with optimizing $\|S(z)\|_\infty, \|T(z)\|_2, \|S(z)\|_2$ in the sense that optimizing one may make the other one far from being optimized.

5. Stabilization of multirate quantized control systems

In this section, we apply the result in section 4 to the case of multirate quantized control systems. The problem setup is the same as shown in Figure 1 except that now the network channels are specifically composed of quantizers. The time quantization is just sampling. For the spatial quantization, both the commonly used logarithmic quantizer given by (1) and the alternative logarithmic quantizer given by (2) are considered.

For either of the two quantizers, the channel capacity is given by

$$\mathfrak{C}_i = \frac{1}{T_i} \ln \delta_i^{-1} = \frac{1}{T_i} \ln \frac{1+\rho_i}{1-\rho_i}.$$

Here, $\frac{1}{T_i}$ is apparently the time quantization density and $\ln \delta_i^{-1}$ can be considered as a measure of the spatial quantization density. Summing up all the channel capacities gives rise to the total channel capacity $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

Both quantizers are nonlinear, however, the uncertainties associated are static without any dynamics. As we mentioned before, in this case, the inequality (5) and (16) may not be necessary for the stabilization of closed-loop system. Nevertheless, we can apply the sufficiency part of Theorem 1 and Theorem 2 to obtain a sufficient condition for the stabilization of the multirate quantized control systems with each quantizer under channel resource allocation.

Theorem 3. *The multirate quantized control system with either the logarithmic quantizer or the alternative logarithmic quantizer is stabilizable by state feedback under an appropriate channel resource allocation, if* $\mathfrak{C} > H_c(A)$.

Theorem 3 shows a tradeoff between the densities of time quantization and spatial quantization. If the time quantization is finer, i.e., sampling faster, then the spatial quantization can be coarser, vice versa. In [10], this tradeoff has been studied for single-input systems with the logarithmic quantizer under the assumption that the sampling and hold scheme use the same time period. There it has been concluded that for a given sampling interval T, the feedback system can be stabilized if

$$\rho > \frac{e^{T\sum_{\Re(\lambda_i)>0}\lambda_i} - 1}{e^{T\sum_{\Re(\lambda_i)>0}\lambda_i} + 1}.$$

Comparatively, our study is more general. On one hand, multirate sampling-hold scheme is allowed. On the other hand, the tradeoff is studied not only for the logarithmic quantizer case but also for the alternative logarithmic quantizer case. With some simple derivations, we have

$$\rho > \frac{e^{T\sum_{\Re(\lambda_i)>0}\lambda_i} - 1}{e^{T\sum_{\Re(\lambda_i)>0}\lambda_i} + 1} \Leftrightarrow \frac{1+\rho}{1-\rho} > e^{T\sum_{\Re(\lambda_i)>0}\lambda_i} \Leftrightarrow \mathfrak{C} > H_c(A).$$

Therefore, Theorem 3 extends the result in [10].

6. An illustrative example

In this section, we give an example to illustrate how the channel/controller co-design is carried out to stabilize the multirate quantized control system with either the commonly used logarithmic quantizer or the alternative logarithmic quantizer. The advantage of appropriate channel resource allocation is also demonstrated by comparing with the case when inappropriate resource allocation is used.

Consider an unstable continuous-time system $[A|B]$ with

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The initial condition used in the simulation is $x_0 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}'$. Clearly, the system is stabilizable. However, $[A|\alpha_1 B_1 + \alpha_2 B_2]$ is not stabilizable for any $\alpha_1, \alpha_2 \in \mathbb{R}$, since the matrix $\begin{bmatrix} \lambda I - A & \alpha_1 B_1 + \alpha_2 B_2 \end{bmatrix}$ loses row rank when $\lambda = 1$. This means that it is impossible to convert $[A|B]$ to a stabilizable single-input system by a linear combination of the two inputs. Note that $[A|B]$ is already in the Wonham decomposition form with

$$A = \text{diag}\{A_1, A_2\}, \quad b_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}', \quad b_2 = 1,$$

where $A_1 = \text{diag}\{2, 1\}$ and $A_2 = 1$. The topological entropy of the plant is

$$H_c(A) = H_c(A_1) + H_c(A_2) = (2 + 1) + 1 = 3 + 1 = 4.$$

6.1. The logarithmic quantizer case

Let the overall capacity be given by $\mathfrak{C} = 4.02$. Recall that an allocation such that $\mathfrak{C}_1 > H_c(A_1) = 3$ and $\mathfrak{C}_2 > H_c(A_2) = 1$ subject to $\mathfrak{C}_1 + \mathfrak{C}_2 = \mathfrak{C}$ is feasible. Then we first allocate the capacity among the two input channels as $\mathfrak{C}_1 = 3.01, \mathfrak{C}_2 = 1.01$. Let $T_1 = 0.3$(sec) and $T_2 = 0.2$(sec), then the logarithmic quantizers in the two input channels are characterized by $\delta_1 = e^{-\mathfrak{C}_1 T_1} = 0.405$ and $\delta_2 = e^{-\mathfrak{C}_2 T_2} = 0.817$ respectively.

To design the state feedback gain, we discretize the following two continuous-time single-input systems

$$\left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \text{ and } [1|1] \tag{18}$$

with time period T_1 and T_2 respectively. Solving the $\mathcal{H}_\infty$ optimal complementary sensitivity for the two discretized systems yields the optimal feedback gains $f_1 = \begin{bmatrix} -7.758 & 3.23 \end{bmatrix}, f_2 = -3.155$. Let

$$F = \begin{bmatrix} -7.758 & 3.23 & 0 \\ 0 & 0 & -3.155 \end{bmatrix}. \tag{19}$$

With the above co-design of input channels and state feedback gain F, the continuous-time evolution of the plant states is shown in Figure 4. The state converges to zero asymptotically. The quantized control signal is shown in Figure 5.

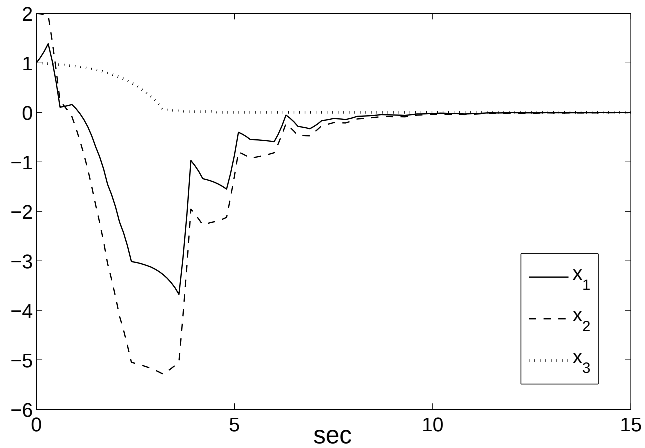

Figure 4: State evolution with logarithmic quantizer under appropriate capacity allocation.

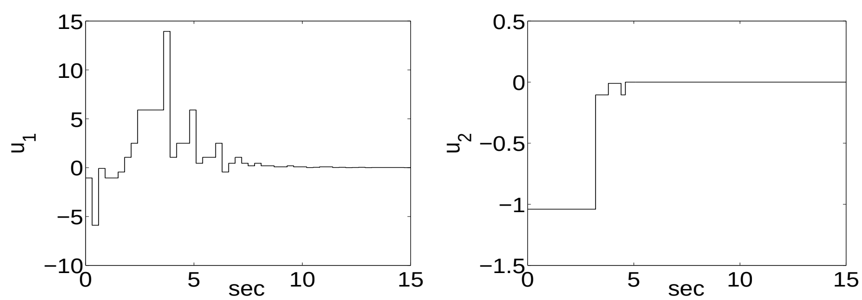

Figure 5: Quantized control signal by logarithmic quantizer.

For comparison, we decrease $\mathfrak{C}_1$ while keeping $\mathfrak{C}_1 + \mathfrak{C}_2$ unchanged. An interesting observation from simulation is that when we allocate $\mathfrak{C}_1$ and $\mathfrak{C}_2$ equally, i.e., $\mathfrak{C}_1 = \mathfrak{C}_2 = 2.01$ and use the sampling period $T_1 = 0.3, T_2 = 0.2$, the closed-loop system with state feedback gain (19) is still stable. This verifies our previous argument that $\mathfrak{C}_1 > H_c(A_1)$ is only sufficient for stabilization since the logarithmic quantizer is static without any dynamics. Now we further decrease $\mathfrak{C}_1$ such that $\mathfrak{C}_1 = 0.6, \mathfrak{C}_2 = 3.42$. As shown in

Figure 6, with the sampling period $T_1 = 0.3, T_2 = 0.2$ and the state feedback gain (19), x_1 and x_2 diverge which illustrates that the capacity allocated to the first input channel is not enough.

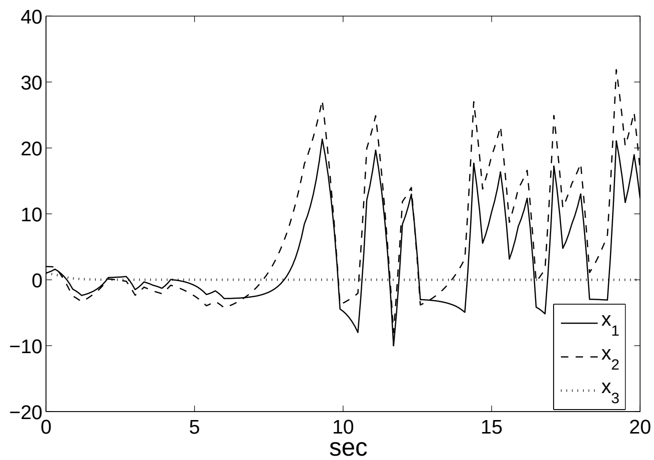

Figure 6: State evolution with logarithmic quantizer under inappropriate capacity allocation.

6.2. The alternative logarithmic quantizer case

Let the overall capacity be again given by $\mathfrak{C} = 4.02$. We first allocate the capacity among the two input channels in an appropriate way as $\mathfrak{C}_1 = 3.01, \mathfrak{C}_2 = 1.01$. Let $T_1 = 0.3$(sec) and $T_2 = 0.2$(sec), then the alternative logarithmic quantizes in the two input channels are characterized by $\delta_1 = e^{-\mathfrak{C}_1 T_1} = 0.405$ and $\delta_2 = e^{-\mathfrak{C}_2 T_2} = 0.817$ respectively. We stress that although δ_1 and δ_2 appear the same as those in the logarithmic quantizer case under the same capacity allocation, the physical meanings are different. The alternative logarithmic quantizer introduces a relative quantization error to the plant while the commonly used logarithmic quantizer introduces a multiplicative quantization error to the plant.

The design of the state feedback gain is also different from that in the logarithmic quantizer case. Solving the $\mathcal{H}_\infty$ optimal sensitivity instead of the $\mathcal{H}_\infty$ optimal complementary sensitivity for the two discretized systems corresponding to (18) yields the optimal feedback gains $f_1 =$

$\begin{bmatrix}-7.092 & 2.953\end{bmatrix}, f_2 = -1.819$. Let

$$F = \begin{bmatrix} -7.092 & 2.953 & 0 \\ 0 & 0 & -1.819 \end{bmatrix}. \quad (20)$$

With this co-design of input channels and state feedback gain F, the continuous-time evolution of the plant states is shown in Figure 7. The state converges to zero asymptotically. The quantized control signal is shown in Figure 8.

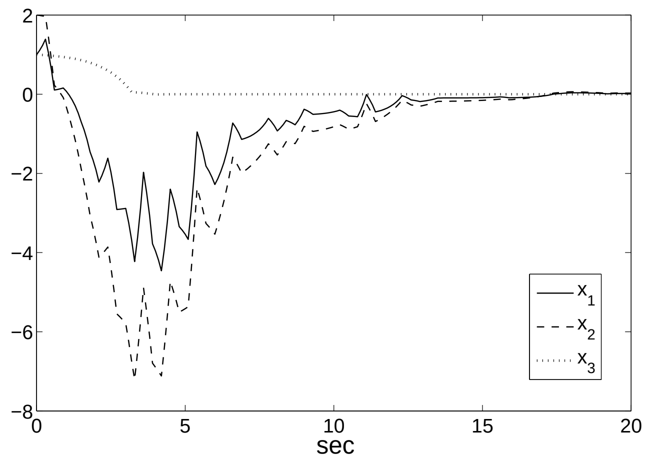

Figure 7: State evolution with alternative logarithmic quantizer under appropriate capacity allocation.

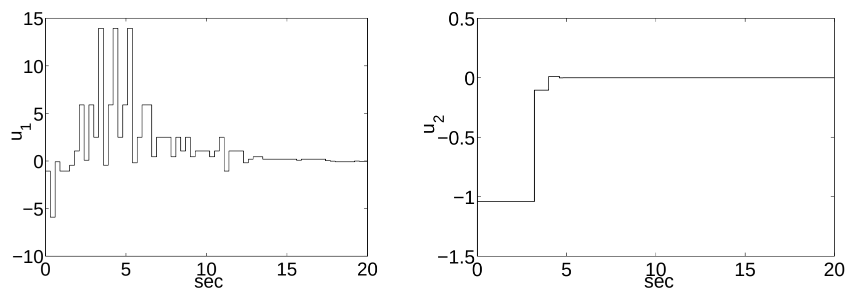

Figure 8: Quantized control signal by alternative logarithmic quantizer.

Similar to the logarithmic quantizer case, here, we make comparison with the case when the capacities are allocated as $\mathfrak{C}_1 = 0.9, \mathfrak{C}_2 = 3.12$. The sampling periods are still $T_1 = 0.3(\text{sec}), T_2 = 0.2(\text{sec})$. As shown in Figure 9, with this allocation and applying the state feedback gain (20), x_1 and x_2 diverge which again illustrates that the capacity allocated to the first input channel is not enough.

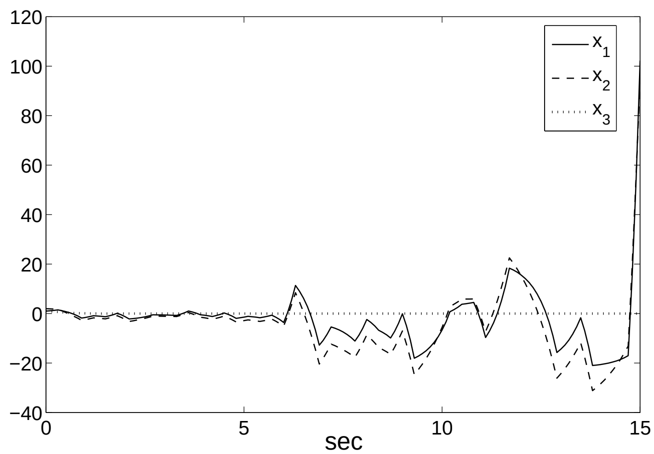

Figure 9: State evolution with alternative logarithmic quantizer under inappropriate capacity allocation.

7. Conclusion

In this paper, we study the stabilization of NCSs with multirate sampling. The input channels are modeled in two different ways, i.e., the SER model and R-SER model. One of the novelties of this paper is that different sampling rates are allowed in different input channels leading to a multirate NCS. With the lifting technique, the stabilization problem of the multirate NCS is transformed to the stabilization problem of the lifted feedback system.

The main contribution of this work is finding the minimum channel capacity required for stabilization by applying channel resource allocation. For a given total channel capacity, the controller designer has the freedom to allocate the capacities among different input channels while designing the controller simultaneously. This channel/controller co-design sheds some light on the trend of integration of the system design and controller design in future engineering applications. By this co-design, we show that for each channel model, the multirate NCS can be stabilized by state feedback under an appropriate resource allocation, if and only if the total channel capacity is greater than the topological entropy of the plant. We also apply the result to multirate quantized control systems. Both the commonly used logarithmic quantizer and the alternative logarithmic quantizer are considered. For each case, a sufficient condition for stabilization is obtained which involves a trade-off between the densities of time quantization and spatial quantization.

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