Stabilization of Networked Multi-Input Systems over AWGN Channels with Channel Resource Allocation

Wei Chen, Li Qiu, and Guoxiang Gu

Abstract— **In this paper, we study stabilization of multi-input networked control systems over additive white Gaussian noise channels. Different from the single-input case, which is available in the literature and boils down to a typical $\mathcal{H}_2$ optimal control problem, the multi-input case involves a judicious allocation of the total capacity among the input channels in addition to the design of the feedback controller. With this channel-controller co-design, we successfully show that a networked multi-input system over additive white Gaussian noise channels can be stabilized by state feedback under channel resource allocation, if and only if the total channel capacity is greater than the topological entropy of the plant. A numerical example is given to demonstrate our result.**

I. Introduction

The networked control systems (NCSs) have received great attention recently. They are feedback systems in which the plant and controller communicate through the shared network. Such systems have many applications, including mobile sensor networks [20], multi-agent systems [19], and automated highway systems [23], etc. Many papers on this topic have been published in technical journals and conferences. See the special issues [1], [2], and the references therein.

One fundamental issue studied in the context of NCS is stabilization under information constraints due to communication channels. These constraints take various forms, such as quantization [9], [13], packet drop [8], data rate constraint [18] and signal-to-noise ratio (SNR) [4] constraint, etc. Numerous results for stabilization of NCSs under information constraints are reported in the literature. For single-input NCSs, logarithmic quantization of the control inputs is considered in [9], [13] which show that the coarsest quantization density ensuring closed-loop stabilizability is given in terms of the Mahler measure of the plant, i.e., the absolute product of the unstable poles. The multiplicative stochastic input channel has been studied in [8] which states that the NCS can be mean-square stabilized by state feedback, if and only if the mean-square capacity of the multiplicative channel exceeds the topological entropy of the plant that is the logarithm of the Mahler measure. For multi-input NCSs, the authors of [14] model the information constraint in the input channels as general sector uncertainties including the logarithmic quantization as a special case. Their main contribution lies in introducing the channel resource allocation to solve the networked stabilization problem. Specifically, they assume that the allowable information constraint is determined by the total network resource available to the channels that can be allocated by the controller designer. Thanks to the additional design freedom gained by the channel resource allocation, an analytical solution has been obtained which states that the largest overall uncertainty bound rendering stabilization is given again in terms of the Mahler measure. In [26], the multi-input NCSs over multiplicative stochastic channels are studied. With the help of channel resource allocation, its authors extend the stabilizability condition in [8] to the multi-input case. These results shed some light on the significance and role of channel resource allocation in NCSs, entailing the idea of channel-controller co-design, i.e., the control designer should participate in the channel design rather than passively taking the given channels. This idea will bring us substantially more freedom and flexibility in designing NCSs, and is envisioned to be common practice in future engineering applications. Later one can see that our main result in this paper can be obtained by allocating the channel resource judiciously.

Another line of work [4], that is most pertinent to our work in this paper, models the information constraint for a single-input NCS as the SNR constraint in an additive white Gaussian noise (AWGN) channel. The technique of $\mathcal{H}_2$ optimal control is used to design the stabilizing controller. A nice analytic solution is obtained for the minimum channel capacity required to stabilize the NCS which is also given in terms of the topological entropy of the plant. Based on the constrained SNR model, [10], [11], [12] have studied further the disturbance attenuation issue. These papers show that the requirement for the channel capacity greater than the topological entropy of the plant remains to be necessary for feedback stabilization, even if nonlinear time-varying communication and control laws are used. One interesting observation from the literature is that the NCS stabilization problem over an AWGN channel is closely related to some nonstandard $\mathcal{H}_2$ optimal control problem. This fact will be seen in this paper when we derive our result later. For the multi-input NCSs over the AWGN channels, unfortunately, the existing results remain to be quite incomplete. An investigation is carried out in [16] which assumes that the total transmission power is constrained

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and can be distributed among different channels, leading to a necessary and sufficient stabilization condition on the transmission power. Different from the result in [4] that is given directly in terms of the topological entropy of the plant, the condition in [16] involves unpleasant computation of the $\mathcal{H}_2$ norm of a transfer function. The latest work in [24] also studies stabilization over power-constrained Gaussian channels. A lower bound on the required transmission power for stabilization is obtained which is not always achievable by LTI encoders and decoders. Motivated by these existing results, we study further stabilization of a multi-input NCS over the AWGN channels in this paper. Instead of assuming the constrained total transmission power, we assume that the total capacity of the input channels are constrained and can be allocated among different channels. By allocating the channel resource, we successfully derive the minimum total capacity required for stabilization given also by the topological entropy of the plant.

The remainder of this paper is organized as follows. Section II formulates the NCS problem to be studied in this paper, and Section III provides some preliminary results on $\mathcal{H}_2$ optimal control. The main result is stated and proved in Section IV. A numerical example is worked out in Section V to illustrate our main result. The paper is concluded in Section VI. The notation of this paper is more or less standard, and will be made clear as we proceed.

II. Problem Formulation

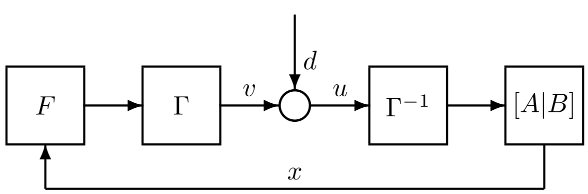

Fig. 1. NCS over AWGN channels.

We consider a discrete-time system described by state-space equation

$$x(k+1) = Ax(k) + Bu(k),$$

where $u(k) \in \mathbb{R}^m$ and $x(k) \in \mathbb{R}^n$. We will denote this system by $[A|B]$ for simplicity. Assume that $[A|B]$ is stabilizable and the state variable $x(k)$ is available for feedback control. For the NCS as shown in Fig. 1, we are interested in stabilizing $[A|B]$ by a constant state feedback controller F over a communication network which is modeled as m parallel AWGN input channels. Here, by parallel, we mean that each component of the controller output is separately sent through an independent AWGN channel to the actuator. Note that we introduce a diagonal scaling matrix Γ with positive diagonal entries:

$$\Gamma = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_m\}.$$

Apparently, increasing γ_i will increase the transmission power in the ith channel. Therefore, the matrix Γ gives us an additional design freedom which enables the possibility to adjust the transmission power in the different input channels. Such a scaling matrix has also been introduced in the literature. See for instance [10], [16], [7].

A standard AWGN channel is depicted in Fig. 2, where the transmitted signal v_i and the noise d_i are zero mean Gaussian random processes with variances $\tilde{\sigma}_i^2$ and σ_i^2 respectively. By [5], the SNR of this channel is defined to be

$$\text{SNR}_i = \frac{\tilde{\sigma}_i^2}{\sigma_i^2}, \tag{1}$$

and the channel capacity is

$$\mathfrak{C}_i = \frac{1}{2}\log(1 + \text{SNR}_i).$$

The total capacity of the input channels is then given by

$$\mathfrak{C} = \mathfrak{C}_1 + \cdots + \mathfrak{C}_m.$$

Clearly, the larger capacity, or equivalently the larger SNR, implies that more reliable information can be transmitted through the channel. Therefore, the capacity $\mathfrak{C}_i$ measures the information constraint of the ith channel and the total capacity $\mathfrak{C}$ measures the information constraint of the communication network.

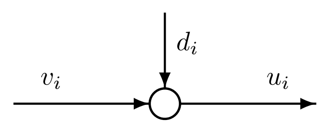

Fig. 2. An AWGN channel.

Assume that all the signals in Fig. 1 are wide sense stationary and the closed-loop system has reached its steady state. According to our setup, the total noise d is a vector white Gaussian noise with covariance

$$\Sigma^2 = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}.$$

The closed-loop transfer function from the noise d to the signal v is the complimentary sensitivity function

$$T(z) = \Gamma F(zI - A - BF)^{-1}B\Gamma^{-1}.$$

Then the power spectrum density of v_i is given by

$$\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii},$$

and the mean power of v_i is

$$\frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega,$$

where $\{\cdot\}_{ii}$ stands for the ith diagonal element of the matrix. In view of (1), the SNR of channel i is expressed as

$$\begin{aligned} \text{SNR}_i &= \frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega / \sigma_i^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \{\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}\}_{ii} d\omega. \end{aligned}$$

Consequently, the capacity of channel i is given by

$$\mathfrak{C}_i = \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii},$$

yielding the total channel capacity

$$\begin{aligned}\mathfrak{C} &= \mathfrak{C}_1 + \cdots + \mathfrak{C}_m \\ &= \frac{1}{2}\log\prod_{i=1}^{m}\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}.\end{aligned}$$

Our objective is to find the smallest total channel capacity such that the NCS over AWGN channels can be stabilized by a constant state feedback controller, i.e., to find

$$\inf_{F:A+BF \text{ is stable}} \mathfrak{C} \tag{2}$$

with given $[A|B]$ and $\gamma_1, \ldots, \gamma_m > 0$. This is a difficult problem. However, by judiciously allocating the channel resource, we are able to mitigate this difficulty and derive the same nice analytic solution as in [4] derived for the single-input case. For this purpose, we assume that the total channel capacity $\mathfrak{C}$ is given and can be allocated among different input channels. The capacity of each channel is determined by the SNR which is proportional to the transmission power. Since the matrix Γ enables the possibility to adjust the transmission power in the different input channels, the total channel capacity can be allocated indirectly here by choosing an appropriate Γ. How to allocate the channel resource appropriately for control of NCS can be considered as a case of channel-controller co-design. The controller designer should simultaneously design the controller and channels to stabilize the closed-loop feedback system. Applying this channel-controller co-design gives rise to the following minimization problem

$$\inf_{\gamma_1,\ldots,\gamma_m>0}\ \inf_{F:A+BF \text{ is stable}} \mathfrak{C} \tag{3}$$

that is the infimum of the total channel capacity required for networked stabilization with channel resource allocation. At first sight, this problem looks even harder than problem (2). However, surprisingly, it can be analytically solved, as shown in the remainder of this paper.

Before proceeding, let us recall two notions which were introduced to dynamical systems theory long time ago but only appeared in the control literature recently. One is the Mahler measure [17] of an $n\times n$ matrix A, denoted by $M(A)$, which is simply the absolute value of the product of the unstable eigenvalues of A, i.e., $M(A) = \prod_{i=1}^{n}\max\{1, |\lambda_i(A)|\}$. The second is the topological entropy [3] of A, denoted by $h(A)$, which is simply the logarithm of $M(A)$, i.e., $h(A) = \log M(A)$.

III. Preliminary on $\mathcal{H}_2$ optimal control

As discussed in the previous section, the NCS stabilization problem over AWGN channels is closely related to some nonstandard $\mathcal{H}_2$ optimal control problem. Denote $\overline{T}(z) = F(zI - A - BF)^{-1}$. The following lemma studies

$$\Omega = \left\{\frac{1}{2\pi}\int_0^{2\pi}\overline{T}(e^{j\omega})^*\overline{T}(e^{j\omega})d\omega \colon A + BF \text{ is stable}\right\},$$

which is a subset of the partially ordered set (poset) of $n\times n$ positive semi-definite matrices. We briefly review several concepts [6] in the theory of poset. The infimum of Ω, denoted as $\inf\Omega$, is the greatest lower bound of Ω. The least element of Ω, if exists, is an element of Ω which is less than or equal to any other element of Ω. Apparently, the subset Ω contains a least element if and only if $\inf\Omega \in \Omega$. Denote the closure of Ω by $\overline{\Omega}$.

Lemma 1: Let $[A|B]$ be stabilizable. Then $\inf\Omega \in \overline{\Omega}$.

Proof: We first consider the case when A has no eigenvalues on the unit circle. By the Parseval's identity [21], we have

$$\frac{1}{2\pi}\int_0^{2\pi}\overline{T}(e^{j\omega})^*\overline{T}(e^{j\omega})d\omega = \sum_{k=0}^{\infty}(A+BF)'^k F'F(A+BF)^k.$$

The right-hand side of the above equation is precisely the solution to

$$P = (A + BF)'P(A + BF) + F'F$$

that is a discrete-time Lyapunov equation. This fact implies that

$$\Omega = \{P : P = (A+BF)'P(A+BF)+F'F, A+BF \text{ is stable}\}.$$

It is well known from the $\mathcal{H}_2$ optimal control theory [22] that $\inf\Omega = X$, where X is the unique stabilizing solution to the algebraic Riccati equation (ARE)

$$A'X(I + BB'X)^{-1}A = X. \tag{4}$$

The corresponding optimal gain F is given by

$$F = -B'X(I + BB'X)^{-1}A. \tag{5}$$

Moreover, we have $\inf\Omega = X \in \Omega$, which implies that X is in fact the least element of Ω . This completes the proof for the case when A has no eigenvalues on the unit circle.

If A has eigenvalues on the unit circle, the desired feedback gain (5) cannot be achieved. Therefore, the least element of Ω does not exist in this case. Nevertheless, we can let $A_\epsilon = (1 + \epsilon)A$ with $\epsilon > 0$ such that A_ϵ has the same number of eigenvalues inside the unit circle as A but no eigenvalues on the unit circle. We also define the subset Ω_ϵ correspondingly. Applying the above derivation to system $[A_\epsilon|B]$ yields that Ω_ϵ has a least element given by the stabilizing solution X_ϵ to ARE

$$A_\epsilon' X_\epsilon(I + BB'X_\epsilon)^{-1}A_\epsilon = X_\epsilon.$$

Taking the limit $\epsilon \to 0$, we get $\lim_{\epsilon\to 0} X_\epsilon = X$, where X is the unique semi-stabilizing solution to (4) in the sense that all the eigenvalues of $A - BB'X(I + BB'X)^{-1}A$ lie in the closed unit disk. This implies that $\inf\Omega \in \overline{\Omega}$ which concludes the proof. ■

Remark 1: The proof of Lemma 1 implies that the eigenvalues of A on the unit circle have no effect on the infimum of $\mathcal{H}_2$ norm of the complementary sensitivity function $T(z)$. In addition, the system $[A|B]$ can be assumed to have decomposition $[A|B] = \left[\begin{array}{cc|c} A_{\rm s} & 0 & B_{\rm s} \\ 0 & A_{\rm u} & B_{\rm u}\end{array}\right]$, where $A_{\rm s}$ is stable and A_u is unstable. By decomposing F into $\begin{bmatrix} F_s & F_u \end{bmatrix}$ with compatible dimensions, $F_s = 0$ can be used in minimizing the $\mathcal{H}_2$ norm of $T(z)$. As a result, the stable eigenvalues of A also have no effect on the optimization value. Therefore, we can simply assume that A is anti-stable without loss of generality when we encounter optimization of $T(z)$ in the sequel.

The following corollary can be easily deduced from Lemma 1.

Corollary 1: Let $[A|B]$ be stabilizable. Then

$$\inf_{F: A+BF \text{ is stable}} \frac{1}{2} \log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega}) d\omega\right) = h(A). \tag{6}$$

Proof: By Remark 1, we only need to consider the case when A is anti-stable. Applying Lemma 1 together with the fact that the log determinant function is operator monotone increasing on the cone of positive definite matrices yields

$$\inf_{F: A+BF \text{ is stable}} \frac{1}{2} \log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega}) d\omega\right) = \frac{1}{2}\log\det(I + \Gamma^{-1}B'XB\Gamma^{-1}), \tag{7}$$

where X is the unique stabilizing solution to the ARE

$$A'X(I + B\Gamma^{-2}B'X)^{-1}A = X.$$

Moreover, $X > 0$ and has a closed form expression

$$X = \left(\sum_{k=1}^{\infty} A^{-k}B\Gamma^{-2}B'A'^{-k}\right)^{-1}.$$

Therefore,

$$\begin{aligned}\det(I + \Gamma^{-1}B'XB\Gamma^{-1}) &= \det(I + B\Gamma^{-2}B'X) \\ &= \det(X^{-1}A'XA) = M(A)^2.\end{aligned}$$

The above equality together with (7) leads to (6) which concludes the proof. ■

In our application, we are more interested in a performance index with the order of $T(e^{j\omega})$ and $T(e^{j\omega})^*$ in (6) reversed, as shown in the following corollary.

Corollary 2: Let $[A|B]$ be stabilizable. Then

$$\inf_{F: A+BF \text{ is stable}} \frac{1}{2} \log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega}) T(e^{j\omega})^* d\omega\right) \geq h(A). \tag{8}$$

Proof: For an arbitrary F such that $A+BF$ is stable, the matrix $A' + F'B'$ is also stable. This implies that the system $[A'|F']$ is stabilizable. Moreover, B' is a stabilizing state feedback gain. In this case, the complementary sensitivity function corresponding to $[A'|F']$ is $T'(z) = \Gamma^{-1}B'(zI - A' - F'B')^{-1}F'\Gamma$. According to Corollary 1,

$$\begin{aligned}&\frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T'(e^{j\omega})^* T'(e^{j\omega}) d\omega\right) \\ &= \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{-j\omega}) T(e^{-j\omega})^* d\omega\right) \\ &= \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega}) T(e^{j\omega})^* d\omega\right) \geq h(A).\end{aligned}$$

Since the choice of stabilizing F is arbitrary, the inequality (8) follows which concludes the proof. ■

One can observe that when $T(e^{j\omega})$ is normal, i.e., $T(e^{j\omega})T(e^{j\omega})^* = T(e^{j\omega})^*T(e^{j\omega})$ for all $\omega \in [0, 2\pi)$, the left-hand side of (8) is the same as that of (6), therefore the equality in (8) holds. It is natural to ask whether the equality holds in general. At this moment, we are not sure about this. Nevertheless, our guess is that the answer is negative.

In the single-input case, the left-hand sides of (6) and (8) are the same and they are equivalent to a standard $\mathcal{H}_2$ optimization problem, which has been studied in some other places, for instance, [4], [8].

Corollary 3: Let $[A|B]$ be stabilizable and $m = 1$. Then

$$\inf_{F: A+BF \text{ is stable}} \|T(z)\|_2 = [M(A)^2 - 1]^{1/2}.$$

Proof: This corollary follows from Corollary 1. ■

Before moving on to the next section, we briefly review another useful technique called Wonham decomposition. It was originally put forward in [25] to solve the multi-input pole placement problem. Given a stabilizable multi-input system $[A|B]$, we can carry out the controllable-uncontrollable decomposition with respect to the first column of B by a similarity transformation such that $[A|B]$ is equivalent to

$$\left[\begin{bmatrix} A_1 & * \\ 0 & \tilde{A}_2 \end{bmatrix}\middle|\begin{bmatrix} b_1 & * \\ 0 & \tilde{B}_2 \end{bmatrix}\right].$$

Then we proceed to do the controllable-uncontrollable decomposition to the system $[\tilde{A}_2|\tilde{B}_2]$ with respect to the first column of $\tilde{B}_2$. Continuing this process yields the following Wonham decomposition

$$\left[\begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix}\middle|\begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}\right] \tag{9}$$

that is equivalent to $[A|B]$, where each pair $[A_i|b_i]$ is stabilizable.

IV. Main Result

The main result of this paper is presented in the following theorem.

Theorem 1: Let $[A|B]$ be stabilizable. Then the multi-input NCS over AWGN channels can be stabilized by state feedback under channel resource allocation, if and only if $\mathfrak{C} > h(A)$.

Proof: We only need to show

$$\inf_{\gamma_1,\ldots,\gamma_m > 0} \inf_{F: A+BF \text{ is stable}} \mathfrak{C} = h(A).$$

In light of Remark 1, we can simply assume that A is anti-stable. We first prove that for a given stabilizing state feedback gain F and a scaling matrix Γ, the total channel capacity $\mathfrak{C} \geq h(A)$. Denote $\tilde{B} = B\Gamma^{-1}\Sigma$ and $\tilde{F} = \Sigma^{-1}\Gamma F$, then $[A|\tilde{B}]$ is stabilizable and $\tilde{F}$ is a stabilizing gain for this system. Let $\tilde{T}(z)=\tilde{F}(zI-A-\tilde{B}\tilde{F})^{-1}\tilde{B}$. By Corollary 2, we have

$$\begin{aligned}&\frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^*d\omega\right)\\&\quad=\frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2T(e^{j\omega})^*\Sigma^{-1}d\omega\right)\\&\quad\geq h(A).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathfrak{C}&=\frac{1}{2}\log\prod_{i=1}^{m}\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}\\&\geq\frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2T(e^{j\omega})^*\Sigma^{-1}d\omega\right)\\&\geq h(A),\end{aligned}$$

where the first inequality follows from Hadamard's inequality [15]: for any $m\times m$ positive definite matrix $Q=[q_{ij}]$, it holds $\det(Q)\leq\Pi_{i=1}^m q_{ii}$ and the equality holds if and only if Q is diagonal.

Without loss of generality, $[A|B]$ is assumed to have the Wonham decomposition given by (9), where each pair $[A_i|b_i]$ is stabilizable with state dimension n_i. Now we show that for any $\epsilon>0$, if the total capacity constraint is given by $h(A)+\epsilon$, then one can find an allocation of this constraint among the input channels in the form $\{h(A_1)+\frac{\epsilon}{m},\ldots,h(A_m)+\frac{\epsilon}{m}\}$ and simultaneously design a feedback gain F such that the closed-loop system is stable and each channel capacity satisfies the constraint $\mathfrak{C}_i<h(A_i)+\frac{\epsilon}{m}$. The allocation of channel capacities is done indirectly here by choosing an appropriate scaling matrix Γ. Specifically, let

$$\Gamma^{-1}\Sigma=\mathrm{diag}\{1,\delta,\ldots,\delta^{m-1}\}$$

with δ a small positive real number. Define

$$P=\mathrm{diag}\{I_{n_1},\delta I_{n_2},\ldots,\delta^{m-1}I_{n_m}\}.$$

Then

$$\begin{aligned}\tilde{T}(z)&=\tilde{F}(zI-A-\tilde{B}\tilde{F})^{-1}\tilde{B}\\&=\tilde{F}P(zI-P^{-1}AP-P^{-1}\tilde{B}\tilde{F}P)^{-1}P^{-1}\tilde{B},\end{aligned}$$

where

$$P^{-1}AP=\begin{bmatrix}A_1&o(\delta)&\cdots&o(\delta)\\0&A_2&\ddots&\vdots\\\vdots&\ddots&\ddots&o(\delta)\\0&\cdots&0&A_m\end{bmatrix},$$

$$P^{-1}\tilde{B}=\begin{bmatrix}b_1&o(\delta)&\cdots&o(\delta)\\0&b_2&\ddots&\vdots\\\vdots&\ddots&\ddots&o(\delta)\\0&\cdots&0&b_m\end{bmatrix},$$

and $\frac{o(\delta)}{\delta}$ approaches to a finite constant as $\delta\to0$.

For any given total capacity constraint $h(A)+\epsilon$, we can always find an allocation of the total constraint in the form $\{h(A_1)+\frac{\epsilon}{m},\ldots,h(A_m)+\frac{\epsilon}{m}\}$. By Corollary 3, for each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|T_i(z)\|_2^2=M(A_i)^2-1$, where $T_i(z)=f_i(zI-A_i-b_i)^{-1}b_i$. Now let $F=\tilde{F}P=\mathrm{diag}\{f_1,f_2,\ldots,f_m\}$, then

$$\begin{aligned}\mathfrak{C}_i&=\frac{1}{2}\log\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}\\&=\frac{1}{2}\log\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^*d\omega\right\}_{ii}\\&=\frac{1}{2}\log\left(1+\|T_i(z)\|_2^2\right)+o(\delta)\\&=\frac{1}{2}\log M(A_i)^2+o(\delta)\\&=h(A_i)+o(\delta).\end{aligned}$$

By choosing a sufficiently small $\delta>0$, the actual channel capacities can be made to satisfy the constraints $\mathfrak{C}_i<h(A_i)+\frac{\epsilon}{m}$ for $i=1,\ldots,m$. Apparently, the total capacity satisfies $\mathfrak{C}<h(A)+\epsilon$. ∎

Theorem 1 solves the problem as formulated in (3), and provides a necessary and sufficient condition for stabilization of the multi-input NCS over AWGN channels with the help of channel resource allocation. The minimum total channel capacity required for stabilization is equal to the topological entropy of the plant that is the same as that needed for the single-input case. We want to emphasize that the channel capacity allocation is done indirectly here by choosing the scaling matrix Γ, i.e., by adjusting the transmission power in the different input channels. The difference from the setup in [16], [24] lies in that the total channel capacity, rather than the total transmission power, is assumed to be constrained. Once again, we witness the benefits brought in by the channel-controller co-design. With the additional design freedom gained by the channel resource allocation, the problem of networked stabilization becomes well formulated and admits a nice analytic solution.

V. An Illustrative Example

In this section, we provide an example to illustrate the result in Section IV. For the sake of numerical computation, we take the logarithm with base 2 in our example.

Consider an unstable system $[A|B]$ with

$$A=\begin{bmatrix}8&0&0\\0&4&0\\0&0&4\end{bmatrix},\quad B=\begin{bmatrix}B_1&B_2\end{bmatrix}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}.$$

Clearly, $[A|B]$ is stabilizable. However, $[A|\alpha_1B_1+\alpha_2B_2]$ is not stabilizable for any $\alpha_1,\alpha_2\in\mathbb{R}$, since the matrix $\begin{bmatrix}\lambda I-A&\alpha_1B_1+\alpha_2B_2\end{bmatrix}$ loses row rank when $\lambda=4$. This fact implies that it is impossible to convert $[A|B]$ to a stabilizable single-input system by a linear combination of the two inputs. The topological entropy of the plant is

$$h(A)=\log_2 8+\log_2 4+\log_2 4=7.$$

As mentioned before, the channel resource allocation in this case is done by choosing the scaling matrix Γ. Specifically, let

$$\Gamma^{-1}\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}.$$

To design the state feedback gain, we solve the $\mathcal{H}_2$ optimal $T(z)$ for the following two single-input systems:

$$\left[\begin{array}{cc|c} 8 & 0 & 1 \\ 0 & 4 & 1 \end{array}\right] \text{ and } \left[4\,|\,1\right].$$

The optimal state feedback gains are given by $f_1 = \begin{bmatrix} -15.258 & 3.633 \end{bmatrix}$ and $f_2 = -3.75$, respectively. Let

$$F = \begin{bmatrix} -15.258 & 3.633 & 0 \\ 0 & 0 & -3.75 \end{bmatrix}.$$

Under this feedback controller, the numerical results on the channel capacities for different choices of δ are summarized in Table I.

TABLE I

SIMULATION RESULTS.

δ	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}$
10^{-1}	5.1	2	7.1
10^{-2}	$5+1\times10^{-3}$	2	$7+1\times10^{-3}$
10^{-3}	$5+1\times10^{-5}$	2	$7+1\times10^{-5}$

We can see that as $\delta \to 0$, the total capacity $\mathfrak{C} \to h(A)$. In other words, for any $\epsilon > 0$, when the total channel capacity constraint is given by $h(A)+\epsilon$, we can always simultaneously design the state feedback gain F and find an allocation of the capacities among input channels to stabilize the closed-loop system. To demonstrate more clearly how the channel resource allocation is done, let the total capacity constraint be specifically given by $7+4\times10^{-3}$. Then we allocate this constraint among the two input channels as $\{5+2\times10^{-3}, 2+2\times10^{-3}\}$. Now we choose $\delta = 10^{-2}$ and use the state feedback gain F designed above. Under this channel-controller co-design, the channel capacities $\mathfrak{C}_1 = 5+1\times10^{-3} < 5+2\times10^{-3}, \mathfrak{C}_2 = 2 < 2+2\times10^{-3}$ as shown in Table I. The total capacity satisfies the constraint $\mathfrak{C} = 7+1\times10^{-3} < h(A)+\epsilon$.

VI. CONCLUSION

In this paper, we study stabilization of multi-input NCS over AWGN channels. The key idea of our approach is the channel resource allocation. By properly choosing the scaling matrix Γ, the total channel capacity can be allocated indirectly among different input channels. The channel resource allocation, together with a simultaneous design of the feedback gain, consists of a channel-controller co-design. With this co-design, we obtain the minimum total channel capacity required for stabilization of a multi-input NCS over AWGN channels given by the topological entropy of the plant, which is the same as that needed for the single-input case. A numerical example is given to demonstrate our result.

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