Stabilization of Networked Multi-input Systems over a Shared Bus with Scheduling/Control Co-design

Wei Chen, Jing Yao, and Li Qiu

Abstract— **In this paper, we study the stabilization of a continuous-time networked multi-input system over a shared communication bus modeled as a fading channel. Transmission scheduling of the control inputs has to be performed so that only one input signal is transmitted through the channel at one time. We aim at finding the minimum channel capacity rendering state feedback stabilization possible. The main novelty of this work lies in the idea of scheduling/control co-design which suggests that the transmission scheduling should be designed simultaneously with the controller design. By virtue of such co-design, a nice analytic solution is obtained for the minimum channel capacity required for stabilization given in terms of the topological entropy of the plant. A numerical example is provided to illustrate how the scheduling/control co-design is carried out to stabilize the networked system.**

I. Introduction

The networked control systems (NCSs) wherein the feedback loops are closed over communication networks, are gaining more and more popularity in engineering practice. In this work, particular attention is paid to an interesting scenario when the multiple control inputs of an NCS are transmitted through one shared communication bus. We are interested in finding a fundamental limitation on the quality of the communication bus so as to stabilize the NCS.

To better understand the state-of-the-art research on networked stabilization, we briefly review some results in the literature. For discrete-time single-input NCSs, the works in [9], [10] show that the coarsest logarithmic quantization of the control input rendering stabilization possible is given in terms of the Mahler measure of the system, i.e., the absolute product of the unstable poles. The single-input NCS with fading input channel is studied in [8] which states that the NCS can be mean-square (MS) stabilized by state feedback, if and only if the MS channel capacity exceeds the topological entropy of the plant which is the logarithm of the Mahler measure. The networked stabilization over additive white Gaussian noise (AWGN) channel is studied in [3], where the minimum channel capacity rendering stabilization possible for the single-input case is given again in terms of the topological entropy of the plant. For multi-input NCSs, reference [12] introduces the idea of channel/controller co-design, which suggests that the channels and controller should be designed simultaneously to stabilize the system. By virtue of such co-design, a uniform analytic solution for the minimum total channel capacity required for stabilization is obtained for three different channel models, given again in terms of the topological entropy of the plant. The discrete-time multi-input NCSs over parallel fading input channels are studied in [17] which extends the stabilization condition for the single-input case [8] to the multi-input case.

Efforts have also been made to discuss the stabilization of continuous-time NCSs. For example, reference [18] studies the stabilization of continuous-time multi-input systems over parallel fading input channels. With channel/controller co-design, the minimum total channel capacity required for state feedback stabilization is shown to be given by the topological entropy of the plant, i.e., the sum of all the unstable poles. Recently, the work [5] investigates the tradeoff between the densities of time quantization and spatial quantization in the input channels required for stabilization of a continuous-time multi-input NCS.

What drives our effort into this study is the scenario when the multiple control inputs of an NCS have to share a small number of communication channels. This happens frequently in real applications, yet not considered in the above mentioned works [5], [12], [17], [18], since they all assume the same number of communication channels as the number of control inputs. We are particularly interested in the case, as described in the beginning, when there is only one shared communication bus. Transmission scheduling of the control inputs has to be performed so that only one input signal is transmitted through the bus at one time. Such scheduling is reminiscent of the celebrated time-division-multiple-access (TDMA) scheme [19] in the communication theory to avoid collision when a set of clients are transmitting information on a shared bus.

We aim at finding a fundamental limitation on the quality of the communication bus so as to stabilize the NCS. To this end, we propose the idea of scheduling/control co-design, i.e., the transmission scheduling is assumed to be designed simultaneously with the design of the controller. By virtue of this additional design freedom, a nice analytic solution for the minimum channel capacity required for stabilization is obtained, which is again given in terms of the topological entropy of the plant.

Note that the idea of scheduling/control co-design is partially inspired from the channel/controller co-design that

The work described in this paper was partially supported by Hong Kong PhD Fellowship, a grant from the National Natural Science Foundation of China (Project No. 61174158), a grant from China Postdoctoral Science Foundation (Project No. 201104286 and 2012M510117) and a grant from the Research Grants Council of Hong Kong Special Administrative Region, China (Project No. HKUST617912).

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was proposed in [12] and applied in several other works such as [5], [17], [18], etc. The concept of scheduling/control co-design has been used in the study of the embedded control systems for the integration of control and computing [16]. Recently, it attracts interests from the research community of NCSs as well [2], [4], [6].

The remainder of this paper is organized as follows. Section II formulates the networked stabilization problem. Section III provides some preliminary knowledge on MS stabilizability and switched linear systems. The main result is presented in Section IV which gives the minimum channel capacity required for stabilization. A numerical example is given in Section V. Finally, Section VI concludes the paper.

Most notations in this paper are more or less standard and will be made clear as we proceed. The symbol $\odot$ means Hadamard product. Denote the identity matrix by I, the open unit disk by $\mathbb{D}$, and the open left half complex plane by $\mathbb{C}^-$. Denote $\mathcal{S}_n$ as the space of $n \times n$ real symmetric matrices. The spectrum of a linear operator $\mathscr{L}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ is defined to be $\sigma(\mathscr{L}) = \{\lambda \in \mathbb{C} : \mathscr{L}(X) = \lambda X, X \in \mathcal{S}_n, X \neq 0\}$. The MS norm of a transfer function $G(s)$ with dimension $p \times m$, if exists, is defined to be

$$\|G(s)\|_{\mathrm{MS}} = \sqrt{\max_{i=1,2,\ldots,p} \frac{1}{2\pi} \int_{-\infty}^{\infty} [G(j\omega)G'(-j\omega)]_{ii} d\omega}.$$

II. Problem Formulation

Let us start with a continuous-time linear time invariant (LTI) system described by the state space model

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We denote the system as $[A|B]$ for simplicity. Assume that $[A|B]$ is stabilizable and the state $x(t)$ is available for feedback. The traditional control theory assumes ideal transmission of the control inputs to the plant without any errors. It is well known that under this assumption, the system can be stabilized by a static state feedback controller $u(t) = Fx(t)$. However, such state feedback design faces challenges in the network era due to the various information constraints imposed on the imperfect communication channels. The setup of the networked stabilization is shown in Fig. 1.

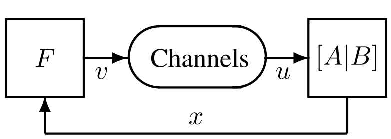

Fig. 1. State feedback via transmission channels.

For the single-input case, by nature, one communication channel suffices to serve the transmission purpose. For the multi-input case, several recent works [5], [12], [17], [18] assume that there are the same number of independent communication channels as the number of the control inputs between the controller and the actuators. Each channel serves one and only one control signal transmission. However, there are applications wherein the input signals have to share a small number of communication channels. In particular, we focus on a simple yet fundamental case when there is only one communication bus serving all the control signals. In this case, a multiplexer and a de-multiplexer have to be used so that the bus can serve the control signals one at a time, as depicted in Fig. 2. The task performed by the multiplexer/de-multiplexer pair is referred to as transmission scheduling. Such a shared bus is beneficial for engineering practice due to the low installment and maintenance cost, especially when the number of the control inputs is quite large.

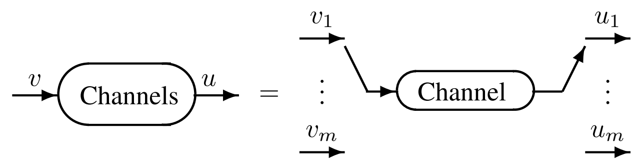

Fig. 2. A shared bus with multiplexer/de-multiplexer pair.

In this work, the shared communication bus is modeled as a fading channel characterized by the following equation:

$$r(t) = \kappa(t)s(t),$$

where $s(t)$ is the transmitted signal, $r(t)$ is the received signal and $\kappa(t)$ is a non-negative white noise process with mean $\boldsymbol{E}[\kappa(t)] = \mu$ and variance $\boldsymbol{E}[(\kappa(t) - \mu)^2] = \sigma^2$. The MS capacity of the channel is defined to be $\mathfrak{C} = \frac{1}{2}\mathrm{SNR}^2 = \frac{1}{2}\frac{\mu^2}{\sigma^2}$. It is clear that the capacity of an ideal channel is infinity. In general, larger capacity indicates that more reliable information can be transmitted through the channel. Therefore, the capacity notion can be considered as a measure of the signal transmission accuracy.

The above NCS over a shared fading channel can be considered as a special type of switched linear system called multiple controller system [13] as below:

$$\dot{x}(t) = Ax(t) + B_\theta \kappa(t) v_\theta(t), \tag{1}$$

where θ is the abbreviation for the switching signal $\theta(t)$ that is piecewise constant and takes values from the index set $\mathcal{I} = \{1, 2, \ldots, m\}$. Assume that $\theta(t) = i$ for $t \in [t_1, t_2)$, then $B_\theta = B_i$ and $v_\theta(t) = F_i x(t)$ for $t \in [t_1, t_2)$, where B_i is the ith column of B and F_i is the ith row of F. The switching signal $\theta(t)$ represents the transmission scheduling of the control inputs and is thus referred to as the scheduling signal in the sequel. A general study for the switched linear systems can be found in [13].

We are interested in finding the minimum channel capacity $\mathfrak{C}$ required for stabilization of the NCS by a static linear state feedback $v(t) = Fx(t)$. Taking the fading effect in the channel into consideration, what we mean by stabilization is in the MS sense that will be clarified later.

Note that periodic scheduling is popular in engineering design and implementation from the practical perspective. Moreover, as stated in Theorem 3.11 in [13], the stabilization of a switched linear system can be accomplished if and only if it can be accomplished with a periodic scheduling signal.

Therefore, hereinafter we consider the case of periodic scheduling in the sense that there exists a period T such that $\theta(t+T)=\theta(t), \forall t \geq 0$. In particular, without loss of generality, assume that the control inputs are sequentially transmitted in the natural order from the first input to the last input. Denote $\pi=\begin{bmatrix}\pi_1 & \pi_2 & \dots & \pi_m\end{bmatrix}'$ as a probability vector, where $0 \leq \pi_i \leq 1, \sum_{i=1}^m \pi_i = 1$. Then a periodic scheduling signal can be expressed as

$$\theta(t)=\begin{cases}1, & \text{if } t\in[kT, kT+\pi_1 T),\\ 2, & \text{if } t\in[kT+\pi_1 T, kT+(\pi_1+\pi_2)T),\\ \vdots & \\ m, & \text{if } t\in[kT+(\sum_{i=1}^{m-1}\pi_i)T, (k+1)T),\end{cases} \tag{2}$$

where $k=0,1,2,\dots$. The vector π is referred to as an allocation vector since it determines the transmission time allocated to each control input during one period T.

If the scheduling signal $\theta(t)$ is fixed a priori, the only design freedom is the state feedback gain F. In this case, unfortunately, it is notoriously hard to find the minimum channel capacity rendering stabilization possible. To mitigate this difficulty, we propose the idea of scheduling/control co-design. Specifically, the scheduling signal $\theta(t)$ is not fixed a priori. Instead, it can be designed simultaneously with the controller. A judicious design of the transmission scheduling will greatly facilitate the controller design. By virtue of this co-design, the networked stabilization problem can be nicely solved, leading to an analytic solution for the minimum channel capacity required for stabilization.

Before proceeding, recall that the topological entropy [1] of a matrix $A \in \mathbb{R}^{n\times n}$ is given by $h(A)=\sum_{|\lambda_i|>1}\ln|\lambda_i|$, where λ_i are the eigenvalues of A. Based on this, we define the topological entropy of a continuous-time system $\dot{x}(t)=Ax(t)$ as $H(A)=h(e^A)=\sum_{\Re(\lambda_i)>0}\lambda_i$, where λ_i are the eigenvalues of A.

III. Preliminary

In this section, some preliminary knowledge on MS stabilizability as well as switched linear systems is presented.

We first define the concept of MS stabilizability under the scheduling/control co-design.

Definition 1: $[A|B]$ is said to be MS stabilizable over a shared fading channel with capacity $\mathfrak{C}$ if there exist a scheduling signal $\theta(t)$ and a state feedback gain F such that for every initial state $x(0)$, $N(t)\triangleq \boldsymbol{E}[x(t)x'(t)]$ is well-defined for any $t>0$ and $\lim_{t\to\infty}N(t)=0$.

As mentioned before, the multi-input NCS over a shared fading channel can be considered as a switched linear system given by (1). With the periodic scheduling signal $\theta(t)$ as in (2), applying Itô's formula [11] to $N(t)$ yields

$$\dot{N}(t)=\mathscr{L}_i(N(t)),\ \text{if } t\in\left[kT+\sum_{l=1}^{i-1}\pi_l T, kT+\sum_{l=1}^{i}\pi_l T\right), \tag{3}$$

where $i=1,2,\dots,m$, $k=0,1,2,\dots$, and $\mathscr{L}_i$ is a linear operator from $\mathcal{S}_n$ to $\mathcal{S}_n$ given by

$$\mathscr{L}_i: X\mapsto (A+\mu B_i F_i)X+X(A+\mu B_i F_i)' + \sigma^2 B_i F_i X F_i' B_i'.$$

Integrating both sides of (3) and discretizing $N(t)$ with period T yields

$$N((k+1)T)=\mathscr{T}(N(kT)), k=0,1,2,\dots,$$

where $\mathscr{T}$ is a linear operator from $\mathcal{S}_n$ to $\mathcal{S}_n$ given by

$$\mathscr{T}=e^{\pi_m\mathscr{L}_m T}e^{\pi_{m-1}\mathscr{L}_{m-1}T}\dots e^{\pi_1\mathscr{L}_1 T}. \tag{4}$$

The operator $\mathscr{T}$ is said to be stable if $\sigma(\mathscr{T})\in\mathbb{D}$. One can easily verify that $N(t)\to 0$ when $t\to\infty$ is equivalent to $N(kT)\to 0$ when $k\to\infty$. Hence, the MS stabilization is accomplished if and only if $\sigma(\mathscr{T})\in\mathbb{D}$. However, it is quite difficult to treat $\mathscr{T}$ directly which is the multiplication of exponentials of linear operators. The reason is clarified as below. Given two linear operators $\mathcal{A}$ and $\mathcal{B}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$. If $\mathcal{A}$ commutes with $\mathcal{B}$, then by the power series representation of $e^{\mathcal{A}}$ and $e^{\mathcal{B}}$, it is easy to see that $e^{\mathcal{A}}e^{\mathcal{B}}=e^{\mathcal{A}+\mathcal{B}}$. Unfortunately, in general, this relationship does not hold.

To tackle this difficulty, a formula from Lie algebra known as the Campbell-Baker-Hausdorff (CBH) formula [7] gives a way to relate the product $e^{\mathcal{A}}e^{\mathcal{B}}$ with the sum $\mathcal{A}+\mathcal{B}$ in the general case. Precisely, the CBH formula goes as follows: There exists $\epsilon>0$ such that for $t\in(-\epsilon,\epsilon)$, there holds

$$e^{\mathcal{A}t}e^{\mathcal{B}t}=e^{(\mathcal{A}+\mathcal{B})t+\frac{1}{2}[\mathcal{A},\mathcal{B}]t^2+\frac{1}{12}([\mathcal{A},[\mathcal{A},\mathcal{B}]]+[\mathcal{B},[\mathcal{B},\mathcal{A}]])t^3+\cdots},$$

where $[\mathcal{A},\mathcal{B}]=\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$ is the commutator product of $\mathcal{A}$ and $\mathcal{B}$. Based on the CBH formula, a method called average method [13] is developed in the switched system theory to study the stabilization of switched linear systems. For the current problem at hand, the following lemma can be easily shown with the CBH formula. The details of the proof are omitted for brevity. Similar results can be found in [13], [14].

Lemma 1: Let $\mathcal{A}_1,\dots,\mathcal{A}_m$ be linear operators from $\mathcal{S}_n$ to $\mathcal{S}_n$. Then there exists $\epsilon>0$ such that when $0<t<\epsilon$, it holds

$$e^{\mathcal{A}_m t}e^{\mathcal{A}_{m-1}t}\dots e^{\mathcal{A}_1 t}=e^{(\sum_{i=1}^m\mathcal{A}_i)t+o(t)},$$

where $\frac{o(t)}{t}\to 0$ as $t\to 0$.

Applying Lemma 1 to the operator $\mathscr{T}$ as in (4) yields

$$\mathscr{T}=e^{(\sum_{i=1}^m\pi_i\mathscr{L}_i)T+o(T)} \tag{5}$$

for sufficiently small T. In this case, one can approximate the logarithm of the operator $\mathscr{T}$ by the product of T and an average operator $\mathscr{L}=\sum_{i=1}^m\pi_i\mathscr{L}_i$. Thus the networked stabilization problem amounts to studying $\mathscr{L}$ that is precisely a linear operator from $\mathcal{S}_n$ to $\mathcal{S}_n$ given by

$$\mathscr{L}: X\mapsto (A+BMF)X+X(A+BMF)' + B(\Sigma^2\odot(FXF'))B',$$

where

$$M = \operatorname{diag}\{\pi_1\mu, \pi_2\mu, \ldots, \pi_m\mu\},$$
$$\Sigma = \operatorname{diag}\{\sqrt{\pi_1}\sigma, \sqrt{\pi_2}\sigma, \ldots, \sqrt{\pi_m}\sigma\}.$$

The operator $\mathscr{L}$ is said to be stable if $\sigma(\mathscr{L}) \in \mathbb{C}^-$. Several criteria in verifying the stability of $\mathscr{L}$ is given in the following lemma. The proof can be referred to [18], [8] and is thus omitted here for brevity.

Lemma 2: The following statements are equivalent:

(a) $\sigma(\mathscr{L}) \in \mathbb{C}^-$.

(b) There exists $X > 0$ and F such that $\mathscr{L}(X) < 0$.

(c) There exists $X > 0$ such that

$$A'X + XA - XBM(\Sigma^2 \odot (B'XB))^{-1}MB'X < 0. \quad (6)$$

(d) It holds

$$\inf_{D\in\mathcal{D}, F: A+BMF \text{ is stable}} \|D^{-1}T(s)D\Phi\|_{\text{MS}} < 1, \quad (7)$$

where $T(s) = F(sI - A - BMF)^{-1}BM$, $\Phi = M^{-1}\Sigma$, and $\mathcal{D}$ is the set of all $m \times m$ positive diagonal matrices.

Note that if the allocation vector π is fixed a priori, the search for the optimal F in the optimization problem (7) is not convex in D. Here, the advantage of the scheduling/control co-design stands out. We assume that the allocation vector π can also be designed. In other words, we can determine the transmission time allocated to each control input during one transmission period. In such a case, the objective becomes to simultaneously design the transmission scheduling and the controller so as to stabilize the NCS. With this scheduling/control co-design, surprisingly, the above non-convex optimization problem becomes manageable, as elaborated in the next section.

IV. Main Result

The following theorem gives the main result of this work, i.e., the minimum channel capacity required for networked stabilization over a shared fading channel.

Theorem 1: $[A|B]$ is MS stabilizable over a shared fading channel with capacity $\mathfrak{C}$ if and only if $\mathfrak{C} > H(A)$.

Proof: Without loss of generality, we assume that all the eigenvalues of A lie in the open right half complex plane. This assumption can be removed following the same arguments as in [3], [5], [12], [17], [18].

We first show the necessity. Assume that there exist a state feedback gain F and a periodic scheduling signal $\theta(t)$ as in (2) such that the MS stabilization is achieved, then $\sigma(\mathscr{T}) \in \mathbb{D}$. In view of Lemma 1, one can always find a sufficiently large positive integer N such that $\mathscr{T} = e^{\mathscr{L}T + o(\frac{T}{N})}$ and thus $\sigma(\mathscr{L}) \in \mathbb{C}^-$. By Lemma 2 (c), there exists $X > 0$ such that the inequality (6) holds. Pre-multiplying and post-multiplying $X^{-\frac{1}{2}}$ on both sides of (6) yields

$$X^{-\frac{1}{2}}A'X^{\frac{1}{2}} + X^{\frac{1}{2}}AX^{-\frac{1}{2}} - X^{\frac{1}{2}}BM(\Sigma^2 \odot (B'XB))^{-1}MB'X^{\frac{1}{2}} < 0.$$

Taking trace for both sides of the above inequality yields

$$\begin{aligned} &\operatorname{tr}(X^{-\frac{1}{2}}A'X^{\frac{1}{2}}) + \operatorname{tr}(X^{\frac{1}{2}}AX^{-\frac{1}{2}}) \\ &\quad - \operatorname{tr}(X^{\frac{1}{2}}BM(\Sigma^2 \odot (B'XB))^{-1}MB'X^{\frac{1}{2}}) \\ =&\operatorname{tr}(A') + \operatorname{tr}(A) - \operatorname{tr}(MB'XBM(\Sigma^2 \odot (B'XB))^{-1}) \\ =&2H(A) - 2\mathfrak{C} < 0, \end{aligned}$$

which completes the proof for the necessity.

To show the other direction, we aim to find a positive diagonal matrix D, a state feedback gain F together with an allocation vector π such that the inequality (7) holds. If that is the case, by Lemma 2, $\sigma(\mathscr{L}) \in \mathbb{C}^-$. Then, in view of (5), one can always choose T sufficiently small to make $\sigma(\mathscr{T}) \in \mathbb{D}$ and thus achieve the MS stabilization. In the sequel, the desired matrices D, F and the allocation vector π are constructed.

Without loss of generality, $[A|B]$ is assumed to be of the form given by the Wonham decomposition [15]

$$\left[\begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}\right],$$

where each subsystem $[A_i|b_i]$ is stabilizable with state dimension n_i. Clearly, we have $\sum_{i=1}^m n_i = n$. For each subsystem $[A_i|b_i]$, it has been shown that [3]

$$\inf_{f_i: A_i + b_i\pi_i\mu f_i \text{ is stable}} \|T_i(s)\|_2^2 = 2H(A_i), \quad (8)$$

where

$$T_i(s) = f_i(sI - A_i - b_i\pi_i\mu f_i)^{-1}b_i\pi_i\mu. \quad (9)$$

We now set

$$D = \operatorname{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\}$$

with ϵ a small positive real number. Also define

$$P = \operatorname{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{m-1}I_{n_m}\}.$$

Then

$$\begin{aligned} &D^{-1}T(s)D\Phi \\ &= D^{-1}F(sI - A - BMF)^{-1}BMD\Phi \\ &= D^{-1}FP(sI - P^{-1}AP - P^{-1}BMFP)^{-1}P^{-1}BMD\Phi \\ &= F(sI - P^{-1}AP - P^{-1}BMDF)^{-1}P^{-1}BMD\Phi. \quad (10) \end{aligned}$$

Simple calculations show that

$$P^{-1}AP = \begin{bmatrix} A_1 & O(\epsilon) & \cdots & O(\epsilon) \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O(\epsilon) \\ 0 & \cdots & 0 & A_m \end{bmatrix}, \quad (11)$$

$$P^{-1}BMD = \begin{bmatrix} b_1\pi_1\mu & O(\epsilon) & \cdots & O(\epsilon) \\ 0 & b_2\pi_2\mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & O(\epsilon) \\ 0 & \cdots & 0 & b_m\pi_m\mu \end{bmatrix}, \quad (12)$$

where $\frac{O(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$. Since $\mathfrak{C} > H(A)$ and $H(A) = \sum_{i=1}^{m} H(A_i)$, we can choose $\pi_i = \frac{H(A_i)}{H(A)}$ satisfying $\sum_{i=1}^{m} \pi_i = 1$ and $\pi_i \mathfrak{C} > H(A_i)$. We now set $F = \mathrm{diag}\{f_1, f_2, \ldots, f_m\}$ such that $A_i + b_i \pi_i \mu f_i$ is stable and $\|T_i(s)\|_2^2 < 2\pi_i \mathfrak{C}$, where $T_i(s)$ is given by (9). The existence of such f_i is guaranteed by (8) and the fact that $\pi_i \mathfrak{C} > H(A_i)$. In view of (10), (11) and (12), it can now be verified that

$$
\begin{aligned}
&D^{-1}T(s)D\Phi \\
&\quad = \mathrm{diag}\{\frac{T_1(s)}{\sqrt{2\pi_1\mathfrak{C}}}, \frac{T_2(s)}{\sqrt{2\pi_2\mathfrak{C}}}, \ldots, \frac{T_m(s)}{\sqrt{2\pi_m\mathfrak{C}}}\} + O(\epsilon; s),
\end{aligned}
$$

where $O(\epsilon; s) \to 0$ as $\epsilon \to 0$. Since $\|T_i(s)\|_2 < \sqrt{2\pi_i \mathfrak{C}}$, it follows that $\|D^{-1}T(s)D\Phi\|_{\mathrm{MS}} < 1$ for sufficiently small ϵ. This completes the proof. ■

Remark 1: Re-examining the above lines of proof reveals that the scaling matrix D approximately decomposes $[A|B]$ into m subsystems $[A_i|b_i]$. The topological entropy of each subsystem can be regraded as a measure of its degree of instability. This implies that a subsystem with larger topological entropy is more unstable and intuitively needs more communication resource to stabilize it. This intuition reflexes into the design of the allocation vector π. A feasible allocation is to make $\pi_i \mathfrak{C} > H(A_i)$ and thus is not unique. Taking $\pi_i = \frac{H(A_i)}{H(A)}$ is only one of the feasible allocations. Such allocation of transmission time shares the same spirit of the channel resource allocation as in [5], [12], [17], [18], where the number of communication channels is assumed to be the same as that of the control inputs. A slight difference is that in those works, the capacities are directly allocated among the input channels subject to a total capacity constraint, while in this work, the communication resource is allocated indirectly by determining the transmission time assigned to each control input.

Remark 2: As shown in the proof of Theorem 1, when the channel capacity is close to the fundamental limitation given by $H(A)$, fast switching must be used to accomplish stabilization. However, fast switching may cause unsavory chattering phenomenon in real applications. To avoid this, one needs to increase the channel capacity. The underlying reason goes as follows: When the capacity is larger, one can design a controller to place the spectrum of the average operator $\mathcal{L}$ more far away to the left of the imaginary axis and, thus, in light of the CBH formula, a larger switching period T can be used while the closed-loop stability is maintained. In the extreme case of an ideal communication channel, no matter how large T is, one can always stabilize the system by designing the controller and scheduling signal appropriately.

V. An Illustrative Example

In this section, we work out an example to illustrate how the scheduling/control co-design is carried out to stabilize the multi-input NCS over a shared communication bus.

Consider the following unstable system $[A|B]$:

$$
A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},
$$

with $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$. Clearly, $[A|B]$ is stabilizable. Moreover, it is already in the Wonham decomposition form with

$$
A = \mathrm{diag}\{A_1, A_2\}, \quad b_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}', \quad b_2 = 1,
$$

where $A_1 = \mathrm{diag}\{2, 1\}$ and $A_2 = 1$. The topological entropy of the plant is

$$
H(A) = H(A_1) + H(A_2) = (2+1) + 1 = 3 + 1 = 4.
$$

The two control inputs are transmitted through a shared fading channel. Let $\mu = 4$, $\sigma^2 = 1.98$. The channel capacity is $\mathfrak{C} = \frac{1}{2}\frac{\mu^2}{\sigma^2} = 4.04$ which is greater than $H(A)$ by one percent. Theorem 1 implies that in this case, the multi-input NCS can be MS stabilized under scheduling/control co-design. One such feasible co-design is carried out as below.

Design the periodic scheduling signal $\theta(t)$ as in (2) with $T = 0.1$ (sec) and $\pi = \begin{bmatrix} \frac{H(A_1)}{H(A)} & \frac{H(A_2)}{H(A)} \end{bmatrix}' = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}'$. For the controller design, we solve the $\mathcal{H}_2$ optimal $T_i(s)$ as in (8) for the following two single-input systems:

$$
[A_1|b_1\pi_1\mu] = \left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right] \text{ and } [A_2|b_2\pi_2\mu] = [1|1],
$$

yielding the optimal feedback gains $f_1 = \begin{bmatrix} -4 & 2 \end{bmatrix}$ and $f_2 = -2$, respectively. Let

$$
F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{13}
$$

With this scheduling/control co-design, the Frobenius norm of the state covariance $N(kT)$ converges to zero asymptotically, as shown in Fig. 3.

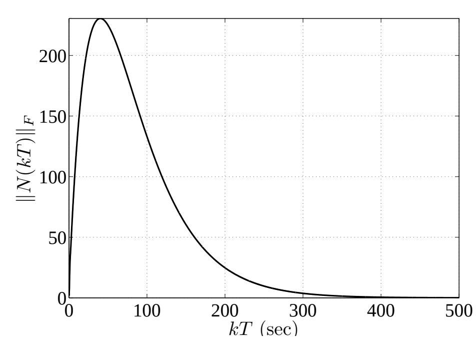

Fig. 3. Evolution of $\|N(kT)\|_F$ when $T = 0.1, \pi_1 = 0.75, \pi_2 = 0.25$.

We stress that the transmission scheduling should be carefully designed, otherwise, the stabilization may not be accomplished no matter what controller is used. To illustrate this point, we first change the allocation vector π to $\begin{bmatrix} 0.74 & 0.26 \end{bmatrix}'$ while keeping $T = 0.1$ (sec) unchanged. In this case, $\pi_1 \mathfrak{C} = 2.99 < H(A_1)$ and thus the allocation vector is infeasible by Remark 1. Indeed, as shown in Fig. 4, with this transmission scheduling and applying the state feedback gain (13), $\|N(kT)\|_F$ diverges due to the infeasible allocation.

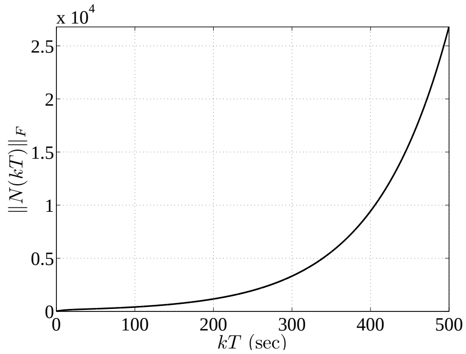

Fig. 4. Evolution of $\|N(kT)\|_F$ when $T = 0.1, \pi_1 = 0.74, \pi_2 = 0.26$.

Now we keep the allocation vector $\pi = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}'$ unchanged and increase the scheduling period T gradually. An interesting observation is that with the increase of T, at first the stabilization can still be accomplished until T reaches ceratin critical value above which the NCS can never be stabilized. This agrees with our previous argument in Remark 2 that fast switching is needed when the channel capacity is quite limited. How to obtain the critical value of T analytically is challenging and needs more investigation. Numerically, it can be easily found with a bisection search. For this example, it is found to be 1.109 (sec). We examine the case when $T = 1.15$ (sec) and the state feedback gain (13) is used. As shown in Fig. 5, $\|N(kT)\|_F$ diverges quickly.

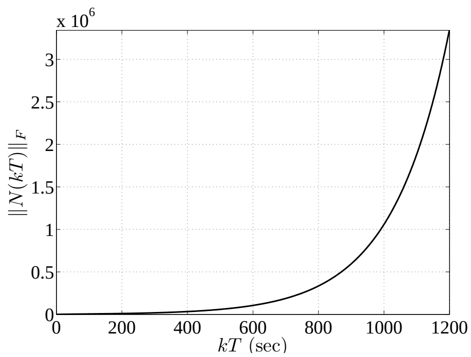

Fig. 5. Evolution of $\|N(kT)\|_F$ when $T = 1.15, \pi_1 = 0.75, \pi_2 = 0.25$.

VI. Conclusion

In this paper, we study the stabilization of a continuous-time networked multi-input system over one shared communication bus modeled as a fading channel. Transmission scheduling of the control inputs has to be performed so that only one input signal is transmitted through the channel at one time. Without loss of generality, periodic scheduling is considered. We aim at finding the minimum channel capacity under which the state feedback stabilization is possible.

The main novelty of this work lies in the idea of scheduling/control co-design which suggests that the transmission scheduling should be designed simultaneously with the controller design. By virtue of such co-design, a nice analytic solution is obtained for the minimum channel capacity required for stabilization given in terms of the topological entropy of the plant. A numerical example is provided to illustrate how the scheduling/control co-design is carried out to stabilize the networked system.

The idea developed here can be extended to more general scenarios when the input signals are transmitted over several shared communication channels. For the sake of practical implementation, this work can also be extended to sampled-data stabilization over shared communication channels. Such extensions are under our current investigation.

Acknowledgment

The authors wish to thank Professor Pravin Varaiya of University of California at Berkeley for helpful suggestions in the formulation of this problem.

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