Stabilization of Networked Multi-Input Systems with Channel Resource Allocation

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***Abstract*—In this paper, we study the problem of state feedback stabilization of a linear time-invariant (LTI) discrete-time multi-input system with imperfect input channels. Each input channel is modeled in three different ways. First it is modeled as an ideal transmission system together with an additive norm bounded uncertainty, introducing a multiplicative uncertainty to the plant. Then it is modeled as an ideal transmission system together with a feedback norm bounded uncertainty, introducing a relative uncertainty to the plant. Finally it is modeled as an additive white Gaussian noise channel. For each of these models, we properly define the capacity of each channel whose sum yields the total capacity of all input channels. We aim at finding the least total channel capacity for stabilization. Different from the single-input case that is available in the literature and boils down to a typical $\mathcal{H}_\infty$ or $\mathcal{H}_2$ optimal control problem, the multi-input case involves allocation of the total capacity among the input channels in addition to the design of the feedback controller. The overall process of channel resource allocation and the controller design can be considered as a case of channel-controller co-design which gives rise to modified nonconvex optimization problems. Surprisingly, the modified nonconvex optimization problems, though appear more complicated, can be solved analytically. The main results of this paper can be summarized into a universal theorem: The state feedback stabilization can be accomplished by the channel-controller co-design, if and only if the total input channel capacity is greater than the topological entropy of the open-loop system.**

***Index Terms*—Networked control system, networked stabilization, Mahler measure, topological entropy, channel resource allocation.**

I. INTRODUCTION

THE networked control systems (NCSs) have received great attention recently. They are feedback systems in which the plant and controller communicate through a shared network. Such systems have wide applications, including mobile sensor networks [35], multi-agent systems [31], and automated highway systems [40], etc. Many papers on this topic have been published in technical journals and conferences. See the special issues [1], [2], and the references therein, as well as the survey papers [20], [33], [21].

A fundamental issue in networked control is stabilization under information constraint in the input channels. Such information constraint takes various forms in different studies, such as data-rate constraint [3], [32], quantization [14], [18], signal-to-noise ratio (SNR) constraint [7], [29], packet drop [43], [13], [46], quantization and packet drop [44], delay [34], [47], etc. For instance, the authors of [14] study stabilization of single-input systems using logarithmic quantized state feedback in the input channel. Based on the Lyapunov function approach, they obtain the coarsest quantization density required for quadratic stabilization in terms of the Mahler measure of the plant, i.e., the absolute product of the unstable poles. The multiplicative stochastic input channel has been studied in [13] which states that the NCS can be mean-square stabilized by state feedback, if and only if the mean-square capacity of the multiplicative channel exceeds the topological entropy of the plant that is the logarithm of the Mahler measure. These results shed some light on the essential role of the Mahler measure, or the topological entropy, that can be considered as a measure of the degree of instability of the open-loop system. Another example supporting this argument can be found in [39], which studies the connections between observability and optimal control via data-rate constrained channels and topological entropy of the plant.

Following the work in [14], the coarsest quantization density has been investigated for multi-input systems in several papers. For instance, a single quantizer is employed in [27], [25] by jointly quantizing the multi-input signals. A logarithmic quantizer is constructed in [27] based on a given control Lyapunov function by quantizing the state space into ellipsoids. In [25], the coarsest quantization is studied with respect to a given control Lyapunov function for a class of multi-input systems that can be stabilized using a one-dimensional subspace of the input space. In [19], instead of a single quantizer, the authors use separate quantizers at different inputs. They utilize a quantization dependent control Lyapunov function to obtain a sufficient condition for stabilization given by linear matrix inequalities (LMIs). While all the above mentioned works on quantized feedback stabilization are based on the Lyapunov function method, a different approach is introduced in [18]. This approach regards the information distortion induced by the logarithmic quantizers in the input channels as sector bounded uncertainties. By using the $\mathcal{H}_\infty$-based robust control technique, the coarsest quantization density required for stabilization in [14] can be recovered for single-input systems.

Another line of work in the literature studies stabilization of NCSs over additive white Gaussian noise (AWGN) channels. For the single-input case, [7] obtains a nice analytic solution for the minimum channel capacity required for stabilization which is again given in terms of the topological entropy of the plant. Based on this work, [15], [16], [17] have studied further the disturbance attenuation issue. These papers show

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that the requirement for the channel capacity greater than the topological entropy of the plant remains to be necessary for feedback stabilization, even if nonlinear time-varying communication and control laws are used. A different investigation is carried out in [29] which studies stabilization of an NCS over parallel power-constrained AWGN output channels via LTI controllers. The minimum total transmission power required for stabilization is given in terms of the $\mathcal{H}_2$ norm of certain transfer function. Finally the latest work in [42] studies state feedback stabilization over power-constrained Gaussian channels. A lower bound on the required transmission power for stabilization is obtained which is not always achievable by LTI encoders and decoders.

Inspired by the existing results discussed above, we investigate state feedback NCS stabilization for multi-input systems. In our setup, the transmission from the controller to the plant input is via non-ideal communication channels. Partial results of this study have been reported in the conference papers [23], [24], [9]. In this paper, the input channels are modeled in three different ways. Firstly, each of them is modeled as an ideal transmission system together with an additive norm bounded uncertainty that is not necessarily a quantizer, neither memoryless nor time-invariant. Although this model is motivated from the logarithmic quantizer studied in [14], [18], it has the potential to capture many other network features such as packet drops and transmission delays. Secondly, each input channel is modeled as an ideal transmission system together with a feedback norm bounded uncertainty, which is motivated from an alternative scheme of logarithmic quantization. Finally, each input channel is modeled as a standard AWGN channel. For each of these models, we properly define the capacity of each channel to measure its information constraint whose sum yields the total channel capacity. Our objective is to find the minimum total channel capacity required for the stabilization of multi-input NCSs. Since each input channel suffers from information distortion, a μ-type control problem arises which is very difficult to solve. To mitigate this difficulty, we introduce the channel resource allocation as a new twist. Instead of imposing the information constraints specified *a priori* as in μ-synthesis, we assume that they are determined by the resource available to the channels which can be allocated by the controller designer subject to a total resource constraint. With this new twist, rather surprisingly, the stabilization problem under each channel model becomes analytically solvable, and the solution is again given in terms of the Mahler measure or topological entropy of the plant as in various studies in [3], [32], [14], [18], [7] for single-input systems.

In this paper, capacity notions are defined for all channel models considered for the convenience of problem formulation and the aesthetics of result statements. They can be regarded as measures of signal transmission accuracy in the channels. However, they are defined in different ways from the Shannon capacity in the information theory [10]. In particular, the capacities for the first and second channel models are defined deterministically while the principle feature of Shannon's information theory is its stochastic foundation. The capacity for the third channel model, an AWGN channel, happens to be the same as the Shannon capacity. It is now well recognized that the Shannon capacity is in general not enough to characterize the information requirement for channels in a feedback system due to the causality constraint in the information processing in a feedback loop. How to define a capacity suitable for channels in a feedback system from an information-theoretic point of view is recently attracting considerable attention in the research community. An attempt is made in [38]. Our definition of capacity for the first two channel models studied in this paper suggests the potential to define capacity in a purely deterministic way.

The remainder of this paper is organized as follows. Section II formulates the networked stabilization problem to be studied in this paper under each of the three channel models. Section III provides some preliminary results on $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal sensitivity and complementary sensitivity. The minimum capacity required for stabilization under each channel model is investigated in Sections IV, V, VI respectively. A numerical example is worked out in Section VII to illustrate our results. The paper is concluded in Section VIII. To highlight the main results, all proofs in Section III are presented in the Appendix. The notation of this paper is more or less standard, and will be made clear as we proceed.

II. Problem Formulation

Consider a discrete-time system described by state-space equation

$$x(k+1) = Ax(k) + Bu(k),$$

where $u(k) \in \mathbb{R}^m$ and $x(k) \in \mathbb{R}^n$. Denote this system by $[A|B]$ for simplicity. Assume that $[A|B]$ is stabilizable and that the state vector $x(k)$ is available for feedback control. We are interested in stabilizing the system by a constant state feedback. Different from the standard setup more than 40 years ago, for instance [45], the signal transmission in the network era is implemented via communication channels. We focus on controller-actuator channels. The new setup is shown in Fig. 1. Parallel transmission strategy is used, i.e., each component v_i of v is transmitted through an independent communication channel.

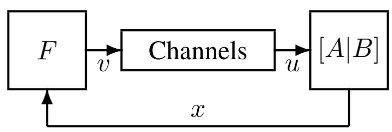

Fig. 1: State feedback via transmission channels.

How a communication channel, especially the one in feedback control, should be modeled is a big issue. There is a vast literature on this issue, and different channel modeling gives rise to a different control method. Three different channel models are considered in this paper, as elaborated in the following.

A. SER model

The first model is motivated by the logarithmic quantization studied in [14] and the realization that it is a sector nonlinearity in [18]. Each channel is modeled as an ideal transmission system with a unity transfer function together with an additive norm bounded uncertainty, as shown in Fig. 2. The uncertainty Δ_i can be a nonlinear, time-varying, and dynamic system. We assume that $\Delta_i(0) = 0$ is the unique equilibrium point and its ℓ_2-induced norm

$$\|\Delta_i\|_\infty = \sup_{v_i \in \ell_2} \frac{\|e_i\|_2}{\|v_i\|_2} \leq \delta_i$$

for some $\delta_i > 0$. In this model, the channel introduces a multiplicative uncertainty to the plant. This uncertainty can be used to model the possible transmission errors due to quantization, signal distortion, as well as other inherent uncertainty in the plant input due to actuator inaccuracy. We define the capacity of channel i as $\mathfrak{C}_i = \log \delta_i^{-1} = -\log \delta_i$. The inverse of the norm bound δ_i^{-1} can be considered as the worst case signal-to-error ratio (SER), since

$$\|\Delta_i\|_\infty^{-1} = \inf_{v_i \in \ell_2} \frac{\|v_i\|_2}{\|e_i\|_2} \geq \delta_i^{-1}.$$

Clearly, larger δ_i indicates that less reliable information can be transmitted through the channel. Therefore, the capacity $\mathfrak{C}_i$ measures properly the information constraint in the ith input channel. Summing up all the capacities $\mathfrak{C}_i$, we obtain the total channel capacity given by $\mathfrak{C} = \mathfrak{C}_1 + \cdots + \mathfrak{C}_m = -\log(\delta_1 \cdots \delta_m)$.

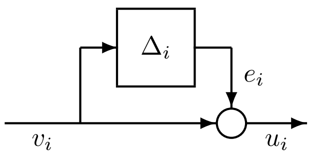

Fig. 2: An SER channel model.

One strong motivation for this channel model is the use of the logarithmic quantizer advocated in [14]. A logarithmic quantizer, depicted in Fig. 3, is defined by the following nonlinear mapping:

$$u_i = Q_{\delta_i}(v_i) := \begin{cases} \rho_i^l \xi_i, & \text{if } \frac{\rho_i^l \xi_i}{1+\delta_i} < v_i \leq \frac{\rho_i^l \xi_i}{1-\delta_i}, \\ 0, & \text{if } v_i = 0, \\ -Q_{\delta_i}(-v_i), & \text{if } v_i < 0, \end{cases}$$

where $\xi_i > 0$, $0 < \rho_i < 1$, $\delta_i = \frac{1-\rho_i}{1+\rho_i}$, and $l = 0, \pm 1, \pm 2, \ldots$. For such a quantizer, the quantization error admits a norm bound

$$\frac{\|u_i - v_i\|_2}{\|v_i\|_2} \leq \delta_i.$$

Apparently, this logarithmic quantizer belongs to the SER channel model described earlier. However, we also stress that our channel model covers not only logarithmic quantization, but also other unknown transmission and actuation errors. One distinction between this SER model and the pure quantization model is that the controller has no way to know the received signal u even though it knows the transmitted signal v exactly.

We are interested in finding the worst channel quality such that the state feedback stabilization is possible. That is, we are interested in finding the minimum possible capacities $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ such that the feedback gain F can be designed

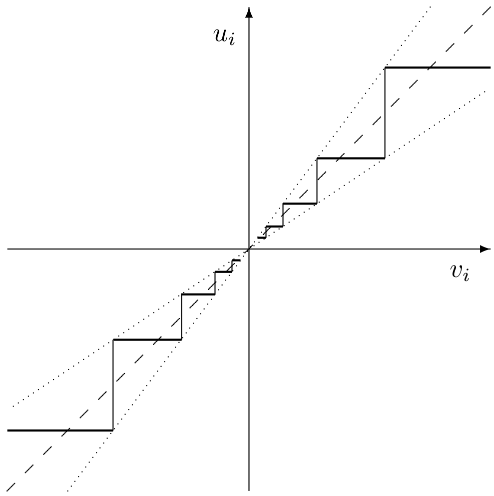

Fig. 3: A logarithmic quantizer.

to stabilize the closed-loop system. When applied to the case of logarithmic quantizer, this problem corresponds to finding the coarsest quantizers so that the state feedback stabilization is possible. When there are several input channels, what it means by minimum capacities or coarsest quantizers needs clarification, and will be made precise later.

B. R-SER model

The second channel model consists of an ideal transmission system with a unity transfer function together with a feedback norm bounded uncertainty [24], as shown in Fig. 4. Again, the uncertainty Δ_i can be a nonlinear, time-varying, and dynamic system. We assume that $\Delta_i(0) = 0$ is the unique equilibrium point and its ℓ_2-induced norm

$$\|\Delta_i\|_\infty = \sup_{u_i \in \ell_2} \frac{\|e_i\|_2}{\|u_i\|_2} \leq \delta_i$$

for some $\delta_i > 0$. In this model, the channel introduces a relative uncertainty to the plant. We define the capacity of channel i as $\mathfrak{C}_i = \log \delta_i^{-1} = -\log \delta_i$ to measure the information constraint in the channel. The inverse of the norm bound δ_i^{-1} can be considered as the worst case received signal-to-error ratio (R-SER), since

$$\|\Delta_i\|_\infty^{-1} = \inf_{u_i \in \ell_2} \frac{\|u_i\|_2}{\|e_i\|_2} \geq \delta_i^{-1}.$$

Larger δ_i indicates that less reliable information can be transmitted through the channel. Summing up all the capacities $\mathfrak{C}_i$, we obtain the total channel capacity given by $\mathfrak{C} = \mathfrak{C}_1 + \cdots + \mathfrak{C}_m = -\log(\delta_1 \cdots \delta_m)$.

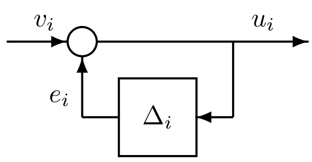

Fig. 4: An R-SER channel model.

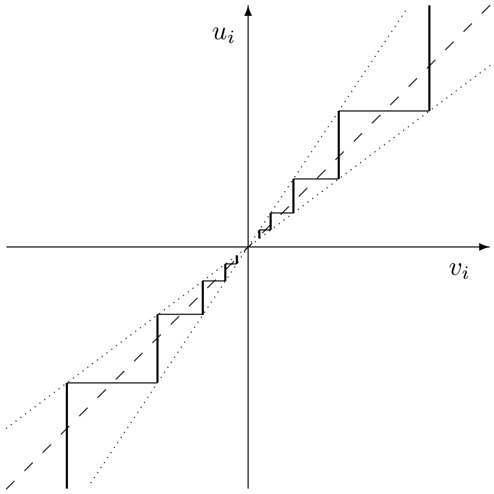

Fig. 5: An alternative logarithmic quantizer.

One strong motivation for this channel model is the use of an alternative scheme of the logarithmic quantizer. The alternative scheme, depicted in Fig. 5, is defined by the following nonlinear mapping:

$$u_i = \tilde{Q}_{\delta_i}(v_i) := \begin{cases} \rho_i^l \xi_i, & \text{if } \rho_i^l \xi_i (1-\delta_i) < v_i \le \rho_i^l \xi_i (1+\delta_i), \\ 0, & \text{if } v_i = 0, \\ -\tilde{Q}_{\delta_i}(-v_i), & \text{if } v_i < 0, \end{cases}$$

where $\xi_i > 0$, $0 < \rho_i < 1$, $\delta_i = \frac{1-\rho_i}{1+\rho_i}$, and $l = 0, \pm 1, \pm 2, \ldots$. A casual look at Fig. 3 and Fig. 5 may not see their difference, but a closer look reveals that the 45° dashed line in Fig. 3, representing the ideal transmission with $u_i = v_i$, halves the vertical segments between the dotted lines, whereas the 45° dashed line in Fig. 5 halves the horizontal segments between the dotted lines. For such an alternative logarithmic quantizer, the quantization error admits a norm bound

$$\frac{\|u_i - v_i\|_2}{\|u_i\|_2} \le \delta_i.$$

Apparently, it belongs to the R-SER channel model described earlier. We advocate the use of this alternative logarithmic quantizer over the commonly used one since it leads to a better optimization problem. This point is a major novelty of this paper and will be justified in details in subsequent sections.

We are again interested in finding the minimum possible $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ such that the feedback gain F can be designed to stabilize the closed-loop system. When applied to the alternative logarithmic quantizer in Fig. 5, it corresponds to finding the coarsest quantizers required for feedback stabilization.

C. SNR model

The third channel model is the standard AWGN channel often used in information theory, as shown in Fig. 6. Here the transmitted signal v_i and the noise d_i are assumed to be zero mean Gaussian random processes. Their variances are assumed to be $\tilde{\sigma}_i^2$ and σ_i^2, respectively. By [10], the SNR of this channel is defined to be

$$\mathrm{SNR}_i = \frac{\tilde{\sigma}_i^2}{\sigma_i^2}, \tag{1}$$

and the channel capacity is $\mathfrak{C}_i = \frac{1}{2}\log(1 + \mathrm{SNR}_i)$. The total capacity of the input channels is given by $\mathfrak{C} = \mathfrak{C}_1 + \cdots + \mathfrak{C}_m$.

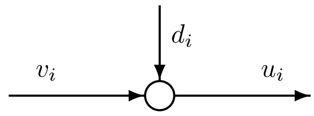

Fig. 6: An AWGN channel.

Clearly, the larger capacity, or equivalently the larger SNR, implies that more reliable information can be transmitted through the channel. Therefore, the capacity $\mathfrak{C}_i$ measures properly the information constraint of the ith channel and the total capacity $\mathfrak{C}$ measures the information constraint of the whole communication network. Again, we are interested in finding the minimum possible $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ such that the feedback gain F can be designed to stabilize the closed-loop system.

III. Background Materials - Optimal sensitivity and complementary sensitivity

Before proceeding, let us recall two concepts which were introduced to dynamical system theory long time ago but only appeared in the control literature recently. One is the Mahler measure [30] of an $n \times n$ matrix A, denoted by $M(A)$, which is simply the absolute value of the product of the unstable eigenvalues of A, i.e., $M(A) = \prod_{i=1}^n \max\{1, |\lambda_i(A)|\}$. The other is the topological entropy [6] of A, denoted by $h(A)$, that is simply the logarithm of $M(A)$, i.e., $h(A) = \log M(A)$.

For the sake of brevity, all proofs in this section are presented in the Appendix.

Consider the feedback system in Fig. 1 and assume that the channels are ideal temporarily. The complementary sensitivity function and sensitivity function at the plant input are

$$\begin{aligned} T(z) &= (I - F(zI - A)^{-1}B)^{-1}F(zI - A)^{-1}B \\ &= F(zI - A - BF)^{-1}B, \\ S(z) &= (I - F(zI - A)^{-1}B)^{-1} \\ &= I + F(zI - A - BF)^{-1}B = I + T(z), \end{aligned}$$

respectively.

To prepare for this section as well as later sections, we first consider a special state feedback $\mathcal{H}_\infty$ control problem in which the objective function is the $\mathcal{H}_\infty$ norm of a scaled convex combination of $S(z)$ and $T(z)$:

$$\|[\theta S(z) + (1-\theta)T(z)]W\|_\infty = \|[T(z) + \theta I]W\|_\infty$$

where $\theta \in [0, 1]$ and $W \in \mathbb{R}^{m \times m}$ is a constant weighting matrix. The discrete-time state feedback $\mathcal{H}_\infty$ control has been studied in the literature [5], [22], [48] which can be specialized to our problem.

Lemma 1: Assume that $[A|B]$ is stabilizable. Then there exists a stabilizing state feedback gain F such that $\|[\theta S(z) + (1-\theta)T(z)]W\|_\infty < 1$, if and only if there exists a stabilizing solution $X \ge 0$ to the following algebraic Riccati equation (ARE):

$$A'X \left[I + B(I - (1-\theta)^2 WW')B'X\right]^{-1} A = X$$

satisfying $W'(\theta^2 I + B'XB)W < I$. If such an $X \geq 0$ exists, then a desired F is given by

$$F = -B'X\left[I + B(I - (1-\theta)^2 WW')B'X\right]^{-1} A.$$

What we are really interested in this paper is the cases when $\theta = 0$ and $\theta = 1$, as shown in the following lemma.

Lemma 2: Assume that A is unstable and $[A|B]$ is stabilizable.

1) If $m = 1$, then

$$\inf_{F \text{ stabilizing}} \|S(z)\|_\infty = \inf_{F \text{ stabilizing}} \|T(z)\|_\infty = M(A).$$

2) If $m > 1$, then

$$\rho(A) \leq \inf_{F \text{ stabilizing}} \|S(z)\|_\infty = \inf_{F \text{ stabilizing}} \|T(z)\|_\infty \leq M(A).$$

We would like to point out that in the case when A is a stable matrix, there hold

$$\inf_{F \text{ stabilizing}} \|T(z)\|_\infty = 0 \quad \text{and} \quad \inf_{F \text{ stabilizing}} \|S(z)\|_\infty = 1.$$

Both infimums are achieved by the trivial optimal feedback gain $F = 0$.

After presenting the $\mathcal{H}_\infty$ optimal values of $S(z)$ and $T(z)$, we now turn our attention to the $\mathcal{H}_2$ optimal values of $S(z)$ and $T(z)$, as shown in the next lemma.

Lemma 3: Assume that $[A|B]$ is stabilizable. Then

$$\inf_{F \text{ stabilizing}} \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega}) d\omega\right) = h(A), \tag{2}$$

$$\inf_{F \text{ stabilizing}} \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega}) T(e^{j\omega})^* d\omega\right) \geq h(A). \tag{3}$$

One can observe that when $T(e^{j\omega})$ is normal, i.e., $T(e^{j\omega})T(e^{j\omega})^* = T(e^{j\omega})^* T(e^{j\omega})$ for all $\omega \in [0, 2\pi)$, the left-hand side of (3) is the same as that of (2), and therefore the equality in (3) holds. It is natural to ask whether the equality holds in general. At this moment, we are not sure about this. Nevertheless, our guess is that the answer is negative.

In the single-input case, the left-hand sides of (2) and (3) are the same and they are equivalent to a standard $\mathcal{H}_2$ optimization problem, which has been studied in some other places, for instance, [13], [7]. Lemma 3 and the fact that $\|S(z)\|_2^2 = \|T(z)\|_2^2 + 1$ immediately result in the following corollary.

Corollary 1: Assume that $[A|B]$ is stabilizable and $m = 1$. Then

$$\inf_{F \text{ stabilizing}} \|T(z)\|_2 = [M(A)^2 - 1]^{1/2},$$
$$\inf_{F \text{ stabilizing}} \|S(z)\|_2 = M(A).$$

Remark 1: The above investigation is mostly concerned with finding the minimal values of the $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ norm of $S(z)$ and $T(z)$. Without loss of generality, assume that A is anti-stable. Then one interesting observation from the proofs in the Appendix is that the minimization of $\|S(z)\|_\infty$, $\|S(z)\|_2$ and $\|T(z)\|_2$ share a common optimal gain F that is given by (29) in the Appendix, where X is the unique stabilizing solution to ARE (23). Moreover,

$$\begin{aligned} A + BF &= A - BB'X(I + BB'X)^{-1}A \\ &= (I + BB'X)^{-1}A = X^{-1}A'^{-1}X. \end{aligned}$$

The above equality indicates that the optimal control law (29) in the Appendix actually moves the unstable poles of the system to their mirror images with respect to the unit circle. As for the minimization of $\|T(z)\|_\infty$, it admits a different optimal control law. This can be seen from ARE (24) whose corresponding feedback gain is different from that of ARE (23). As a consequence, optimizing $\|S(z)\|_\infty$ is preferred to optimizing $\|T(z)\|_\infty$ since the minimization of $\|S(z)\|_\infty$ simultaneously minimizes $\|S(z)\|_2$ and $\|T(z)\|_2$. This fact will become more clear as we proceed.

Before moving on to the next section, we briefly review another useful technique called Wonham decomposition. It was originally put forward in [45] to solve the multi-input pole placement problem. Given a stabilizable multi-input system $[A|B]$, we can carry out the controllable-uncontrollable decomposition with respect to the first column of B by a similarity transformation such that $[A|B]$ is equivalent to

$$\left[\begin{bmatrix} A_1 & * \\ 0 & \tilde{A}_2 \end{bmatrix} \middle| \begin{bmatrix} b_1 & * \\ 0 & \tilde{B}_2 \end{bmatrix}\right].$$

Then we proceed to do the controllable-uncontrollable decomposition to the system $[\tilde{A}_2|\tilde{B}_2]$ with respect to the first column of $\tilde{B}_2$. Continuing this process yields the following Wonham decomposition

$$\left[\begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}\right], \tag{4}$$

that is equivalent to $[A|B]$, where each pair $[A_i|b_i]$ is stabilizable.

IV. Multi-input state feedback stabilization–SER model

Starting from this section, we are dedicated to finding the minimum channel capacity required for stabilization under each of the aforementioned three channel models.

We first study the case when the SER channel model is adopted. In this case, the networked feedback system is shown in Fig. 7, where $\Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_m\}$. Such a diagonal Δ is called structured uncertainty. Handling

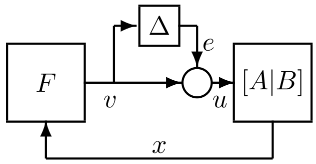

Fig. 7: NCS with SER channel model.

such uncertainties in feedback control involves a μ-synthesis problem which is usually difficult. If the uncertainty bounds $\delta_1, \delta_2, \ldots, \delta_m$ and a stabilizing feedback gain F are given, then the uncertain system is stabilized for all possible uncertainties satisfying the bounds, if and only if [41]

$$\inf_{D\in\mathcal{D}} \|D^{-1}T(z)DD_\delta\|_\infty < 1 \tag{5}$$

where $D_\delta = \operatorname{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ and $\mathcal{D}$ is the set of all $m \times m$ diagonal matrices with positive diagonal entries. The minimization problem in (5) can be converted to a convex problem, and is hence manageable. However the design problem, which aims to find a stabilizing F such that (5) holds, is notoriously hard. This design problem is more or less equivalent to the minimization problem

$$\inf_{F \text{ stabilizing}} \left[\inf_{D\in\mathcal{D}} \|D^{-1}T(z)DD_\delta\|_\infty\right], \tag{6}$$

which cannot be converted to a jointly convex problem.

In networked control, very often the channel capacity $\mathfrak{C}_i$, or equivalently, the SER δ_i^{-1} is associated with certain resource. If we allocate more resource to the ith channel, then we are able to increase its capacity. For example, the use of better and more expensive hardware in the ith channel may increase $\mathfrak{C}_i$; allocation of more communication bandwidth to the ith channel may also increase $\mathfrak{C}_i$. Then in the networked control problem, we may have an overall constraint on the total available resource but we do have the freedom to allocate the resource among different channels. Let us assume that the overall resource constraint is given in terms of $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$. In this case, what we mean by finding the minimum possible capacities in the input channels is to find the minimum total capacity $\mathfrak{C}$ that renders stabilization possible under channel resource allocation. The control problem with channel resource allocation can be considered as a case of channel-controller co-design. The controller designer is in the position to allocate $\mathfrak{C}_i$ optimally among the channels and simultaneously design the feedback gain so that the expression in (6) is minimized. Notice that allocating $\mathfrak{C}_i$ with a given total capacity $\mathfrak{C}$ is equivalent to allocating the error bounds δ_i with a given $\delta = \prod_{i=1}^m \delta_i$. Applying the channel-controller co-design gives rise to a further nested minimization problem: given stabilizable $[A|B]$ and $\delta > 0$, find

$$\inf_{\det D_\delta=\delta} \left\{\inf_{F \text{ stabilizing}} \left[\inf_{D\in\mathcal{D}} \|D^{-1}T(z)DD_\delta\|_\infty\right]\right\}.$$

This problem looks even harder than (6) and is highly nonconvex, but rather surprisingly, it admits a very nice analytic solution.

Theorem 1: Assume that $[A|B]$ is stabilizable. Then the NCS with SER channel model can be stabilized by state feedback under channel resource allocation, if and only if $\mathfrak{C} > h(A)$.

Proof: The condition $\mathfrak{C} > h(A)$ is equivalent to $\inf \mathfrak{C} = h(A)$. We only need to show

$$\inf_{\det D_\delta=\delta} \left\{\inf_{F \text{ stabilizing}} \left[\inf_{D\in\mathcal{D}} \|D^{-1}T(z)DD_\delta\|_\infty\right]\right\} = \begin{cases} 0, & \text{if } A \text{ is stable;} \\ \delta M(A), & \text{if } A \text{ is unstable.} \end{cases}$$

The case when A is stable is trivial. One can just set $F = 0$. For the case when A is unstable, in light of Remark 2 in the Appendix, we assume that A is anti-stable for brevity. We first show that if there exist a stabilizing F and a nonsingular diagonal D such that

$$\|D^{-1}T(z)DD_\delta\|_\infty < 1, \tag{7}$$

then there holds

$$\delta = \prod_{i=1}^m \delta_i < M(A)^{-1}. \tag{8}$$

Rewrite

$$D^{-1}T(z)D = \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}$$

with $\tilde{F} = D^{-1}F$ and $\tilde{B} = BD$. Lemma 1 can be applied to conclude that (7) is equivalent to the existence of $X > 0$ such that

$$X = A'X\left[I + \tilde{B}(I - D_\delta^2)\tilde{B}'X\right]^{-1} A, \tag{9}$$

$$I > D_\delta \tilde{B}' X \tilde{B} D_\delta. \tag{10}$$

Pre-multiplying and post-multiplying both sides of inequality (10) by $\sqrt{D_\delta^{-2} - I}$ yields

$$D_\delta^{-2} - I > \sqrt{I - D_\delta^2}\tilde{B}'X\tilde{B}\sqrt{I - D_\delta^2}.$$

Therefore, if the condition (7) holds, then ARE (9) has a solution $X > 0$ satisfying the above inequality. Together with properties of determinant implies

$$\begin{aligned}\det(D_\delta^{-2}) = \prod_{k=1}^m \delta_k^{-2} &> \det\left(I + \sqrt{I - D_\delta^2}\tilde{B}'X\tilde{B}\sqrt{I - D_\delta^2}\right) \\ &= \det\left(I + \tilde{B}(I - D_\delta^2)\tilde{B}'X\right) = \det(X^{-1}A'XA) \\ &= \det(A')\det(A) = M(A)^2\end{aligned}$$

which verifies inequality (8), completing one direction of the proof.

To show the other direction, we will seek a positive diagonal matrix D, a stabilizing state feedback gain F, and a factorization $\delta = \prod_{i=1}^m \delta_i$ such that (7) holds. Without loss of generality, $[A|B]$ is assumed to have the Wonham decomposition given by (4), where each subsystem $[A_i|b_i]$ is stabilizable with state dimension n_i. We now set

$$D = \operatorname{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\} \tag{11}$$

with ϵ a small positive real number. Also define

$$P = \operatorname{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{m-1} I_{n_m}\}. \tag{12}$$

Then

$$\begin{aligned} &D^{-1}T(z)DD_\delta \\ &\quad= \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}D_\delta \\ &\quad= \tilde{F}P(zI - P^{-1}AP - P^{-1}\tilde{B}\tilde{F}P)^{-1}P^{-1}\tilde{B}D_\delta. \end{aligned}$$

Simple calculations show that

$$P^{-1}AP = \begin{bmatrix} A_1 & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & A_m \end{bmatrix}, \quad (13)$$

$$P^{-1}\tilde{B} = \begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) \\ 0 & \cdots & 0 & b_m \end{bmatrix} \quad (14)$$

and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$. Since

$$\delta < M(A)^{-1} = \prod_{i=1}^{m} M(A_i)^{-1},$$

it is always possible to choose $\delta_i < M(A_i)^{-1}$ such that $\delta = \prod_{i=1}^{m} \delta_i$. We now set $F = \tilde{F}P = \text{diag}\{f_1, f_2, \ldots, f_m\}$ such that $A_i + b_i f_i$ is stable for $1 \leq i \leq m$ and $\|T_i(z)\|_\infty < \delta_i^{-1}$, where $T_i(z) = f_i(zI - A_i - b_i f_i)^{-1} b_i$. Such an f_i exists by the proof of Lemma 2 in the Appendix and the fact that $\delta_i^{-1} > M(A_i)$. It can now be verified that

$$\begin{aligned} &D^{-1}T(z)DD_\delta \\ &\quad = \text{diag}\{T_1(z)\delta_1, T_2(z)\delta_2, \ldots, T_m(z)\delta_m\} + o(\epsilon; z) \end{aligned} \quad (15)$$

where $o(\epsilon; z) \to 0$ as $\epsilon \to 0$ for each $|z| \geq 1$. Since $\|T_i(z)\delta_i\|_\infty < 1$, it follows that $\|D^{-1}T(z)DD_\delta\|_\infty < 1$ for sufficiently small ϵ which concludes the proof. ■

The proof for the sufficiency part is constructive. A closer look at the Wonham decomposition (4) yields that A_i contains all the eigenvalues of A which are controllable by the ith input but not by any of the previous inputs. For a given total capacity $\mathfrak{C} > h(A)$, a feasible allocation of $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ so that $\mathfrak{C} = \sum_{i=1}^{m} \mathfrak{C}_i$ is to make $\mathfrak{C}_i > h(A_i)$. Clearly, such an allocation always exists since $h(A) = \sum_{i=1}^{m} h(A_i)$. To be more precise, $\{\mathfrak{C}_i\}_{i=1}^{m}$ can be allocated as follows: choose $\mathfrak{C}_1$ so that the first input can be used to stabilize all unstable modes controllable from the first input; choose $\mathfrak{C}_2$ so that the second input can be used to stabilize the additional unstable modes controllable from the second input excluding the ones that are already stabilized by the first input; ...; finally $\mathfrak{C}_m$ is chosen to stabilize the remaining unstable modes that are not stabilized by the other inputs. This is exactly the sequential design idea used in the first multi-input pole placement solution in [45]. As for the design of the controller, with the above allocation of $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$, we can separately design a feasible f_i for each subsystem $[A_i|b_i]$ such that $\|T_i(z)\delta_i\|_\infty < 1$. Let $F = \text{diag}\{f_1, f_2, \ldots, f_m\}$. Then for sufficiently small ϵ, $D^{-1}T(z)DD_\delta$ admits an almost block diagonal form given by (15). Consequently, the inequality (7) holds which validates the stabilizing property of such an F.

Notice that if the total capacity $\mathfrak{C} > h(A)$ is not allocated according to $\mathfrak{C}_i > h(A_i)$ for each i, i.e., $\mathfrak{C}_k \leq h(A_k)$ holds for some k, then the networked system can never be stabilized. On the other hand, the re-ordering of inputs does not affect the above scheme, but the sequential design results in a different D and D_δ. Hence, the channel resource allocation and feedback gain design are not unique. However, no matter how the re-ordering is carried out, there is a minimum resource that has to be allocated to the ith channel. This minimum resource is given by the unstable modes only controllable by the input i, which is the minimum stabilization work that input i has to accomplish no matter how the design is carried out.

V. Multi-input state-feedback stabilization – R-SER model

In this section, we study the minimum channel capacity required for state feedback stabilization under the R-SER channel model. The setup is shown in Fig. 8, where $\Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_m\}$.

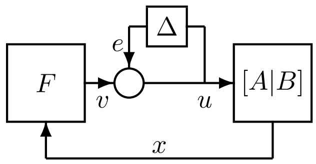

Fig. 8: NCS with R-SER channel model.

Different from the SER model case, here the channels introduce relative uncertainties instead of multiplicative uncertainties to the plant inputs. Because of this difference, stabilization over the R-SER channel model involves optimization of sensitivity instead of complementary sensitivity. To be more precise, for given uncertainty bounds $\delta_1, \delta_2, \ldots, \delta_m$ and a stabilizing feedback gain F, the uncertain system in Fig. 8 is stabilized for all possible uncertainties satisfying the bounds, if and only if

$$\inf_{D \in \mathcal{D}} \|D^{-1}S(z)DD_\delta\|_\infty < 1 \quad (16)$$

where $D_\delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ and $\mathcal{D}$ is the set of all $m \times m$ diagonal matrices with positive diagonal entries. As mentioned before, optimizing $\|S(z)\|_\infty$ is preferred to optimizing $\|T(z)\|_\infty$ since the minimization of $\|S(z)\|_\infty$ simultaneously minimizes $\|S(z)\|_2$ and $\|T(z)\|_2$.

Similar to the SER model case, due to the existence of multiple uncertainties in the loop, a μ-type control problem arises which is very difficult to solve. To overcome this difficulty, again the twist of channel resource allocation is used leading to a channel-controller co-design. We assume that the total channel capacity is given by $\mathfrak{C} = \sum_{i=1}^{m} \mathfrak{C}_i$ and can be allocated among different input channels. This is equivalent to allocating the error bounds δ_i with a given $\delta = \prod_{i=1}^{m} \delta_i$. The channel-controller co-design yields the following minimization problem: given stabilizable $[A|B]$ and $\delta > 0$, find

$$\inf_{\det D_\delta = \delta} \left\{ \inf_{F \text{ stabilizing}} \left[\inf_{D \in \mathcal{D}} \|D^{-1}S(z)DD_\delta\|_\infty \right] \right\}.$$

This problem, again, admits a very nice analytic solution.

Theorem 2: Assume that $[A|B]$ is stabilizable. Then the NCS with R-SER channel model can be stabilized by state feedback under channel resource allocation, if and only if $\mathfrak{C} > h(A)$.

Proof: The condition $\mathfrak{C} > h(A)$ is equivalent to $\inf \mathfrak{C} = h(A)$. We only need to show

$$\inf_{\det D_\delta = \delta} \left\{ \inf_{F \text{ stabilizing}} \left[\inf_{D \in \mathcal{D}} \|D^{-1}S(z)DD_\delta\|_\infty \right] \right\} = \delta M(A).$$

As in the proof of Theorem 1, we assume that A is anti-stable for brevity. We first show that if there exist a stabilizing F and a nonsingular diagonal D such that

$$\|D^{-1}S(z)DD_\delta\|_\infty < 1, \tag{17}$$

then there holds

$$\delta = \prod_{i=1}^{m} \delta_i < M(A)^{-1}. \tag{18}$$

Rewrite

$$D^{-1}S(z)D = I + \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}$$

with $\tilde{F} = D^{-1}F$ and $\tilde{B} = BD$. Lemma 1 can be applied to conclude that (17) is equivalent to the existence of $X > 0$ to

$$\begin{aligned} X &= A'X\left[I + \tilde{B}\tilde{B}'X\right]^{-1} A, \\ I &> D_\delta^2 + D_\delta \tilde{B}'X\tilde{B}D_\delta. \end{aligned} \tag{19}$$

We rewrite inequality (19) as $D_\delta^{-2} > I + \tilde{B}'X\tilde{B}$. It then follows that

$$\begin{aligned} \det(D_\delta^{-2}) = \prod_{k=1}^{m} \delta_k^{-2} &> \det\left(I + \tilde{B}'X\tilde{B}\right) = \det\left(I + \tilde{B}\tilde{B}'X\right) \\ &= \det(X^{-1}A'XA) = \det(A')\det(A) = M(A)^2 \end{aligned}$$

which verifies inequality (18), completing one direction of the proof.

To show the other direction, we will seek a positive diagonal matrix D, a stabilizing state feedback gain F, and a factorization $\delta = \prod_{i=1}^{m} \delta_i$ such that (17) holds. Without loss of generality, $[A|B]$ is assumed to have the Wonham decomposition given by (4), where each subsystem $[A_i|b_i]$ is stabilizable with state dimension n_i. Now choose D as in (11) and define P as in (12). Then

$$\begin{aligned} &D^{-1}S(z)DD_\delta \\ &\quad = I + \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}D_\delta \\ &\quad = I + \tilde{F}P(zI - P^{-1}AP - P^{-1}\tilde{B}\tilde{F}P)^{-1}P^{-1}\tilde{B}D_\delta, \end{aligned}$$

where $P^{-1}AP$ and $P^{-1}\tilde{B}$ are given by (13) and (14) respectively. Since

$$\delta < M(A)^{-1} = \prod_{i=1}^{m} M(A_i)^{-1},$$

it is always possible to choose $\delta_i < M(A_i)^{-1}$ such that $\delta = \prod_{i=1}^{m} \delta_i$. We now set $F = \tilde{F}P = \operatorname{diag}\{f_1, f_2, \ldots, f_m\}$ such that $A_i + b_i f_i$ is stable for all $1 \le i \le m$ and $\|S_i(z)\|_\infty < \delta_i^{-1}$, where $S_i(z) = 1 + f_i(zI - A_i - b_i f_i)^{-1}b_i$. Such an f_i exists by Lemma 2 and the fact that $\delta_i^{-1} > M(A_i)$. It can now be verified that

$$\begin{aligned} &D^{-1}S(z)DD_\delta \\ &\quad = \operatorname{diag}\left\{S_1(z)\delta_1, S_2(z)\delta_2, \ldots, S_m(z)\delta_m\right\} + o(\epsilon; z), \end{aligned}$$

where $o(\epsilon; z) \to 0$ as $\epsilon \to 0$ for each $|z| \ge 1$. Since $\|S_i(z)\delta_i\|_\infty < 1$, it follows that $\|D^{-1}S(z)DD_\delta\|_\infty < 1$ for sufficiently small ϵ which concludes the proof. ■

We want to mention that the remarks following Theorem 1 on how the channel resource allocation is done also apply here.

VI. MULTI-INPUT STATE-FEEDBACK STABILIZATION – SNR MODEL

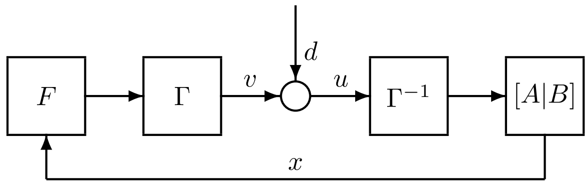

Fig. 9: NCS with SNR channel model.

The same idea extends to the networked state feedback stabilization over the SNR channel model. As shown in Fig. 9, we are interested in stabilizing $[A|B]$ by a constant state feedback controller F over m parallel AWGN input channels. The noise d is a vector white Gaussian noise with covariance $\Sigma^2 = \operatorname{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\}$. Note that we introduce a diagonal scaling matrix $\Gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ with positive diagonal entries. Apparently, increasing γ_i will increase the transmission power in the ith channel. Therefore, the matrix Γ enables the possibility to adjust the transmission power in the different input channels. Since the SNR is proportional to the transmission power and the channel capacity is determined by the SNR, the total channel capacity can be allocated indirectly in this case by choosing an appropriate Γ, which will be elaborated later. Such a scaling matrix has also been introduced in the literature. See for instance [15], [29], [12].

Here, the complimentary sensitivity function that is the closed-loop transfer function from the noise d to the signal v becomes $T(z) = \Gamma F(zI - A - BF)^{-1}B\Gamma^{-1}$. Then the power spectrum density of v_i is given by $\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii}$, and the mean power of v_i is

$$\frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega,$$

where $\{\cdot\}_{ii}$ stands for the ith diagonal element of the matrix. In view of (1), the SNR of channel i is expressed as

$$\begin{aligned} \mathrm{SNR}_i &= \frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega / \sigma_i^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \{\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}\}_{ii} d\omega. \end{aligned}$$

Consequently, the capacity of channel i is given by

$$\mathfrak{C}_i = \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1} d\omega\right\}_{ii},$$

yielding the total channel capacity

$$\begin{aligned} \mathfrak{C} &= \mathfrak{C}_1 + \cdots + \mathfrak{C}_m \\ &= \frac{1}{2}\log\prod_{i=1}^{m}\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1} d\omega\right\}_{ii}. \end{aligned}$$

As before, we are interested in finding the minimum total capacity $\mathfrak{C}$ such that the NCS over AWGN channels can be stabilized by a constant state feedback controller, i.e., to find

$$\inf_{F \text{ stabilizing}} \mathfrak{C} \tag{20}$$

with given $[A|B]$ and $\gamma_1, \ldots, \gamma_m > 0$. This is a difficult problem. However, by applying the channel resource allocation again, we are able to mitigate this difficulty and derive the same nice analytic solution as in [7] obtained for the single-input case. For this purpose, we assume that the total channel capacity $\mathfrak{C}$ is given and can be allocated among different input channels. As we mentioned before, the channel capacity allocation is done indirectly here by choosing an appropriate Γ, which gives rise to the following minimization problem

$$\inf_{\gamma_1,\ldots,\gamma_m>0} \inf_{F \text{ stabilizing}} \mathfrak{C}$$

that is the infimum of the total channel capacity required for networked stabilization with channel resource allocation. At first sight, this problem looks even harder than problem (20). However, surprisingly, it can be analytically solved.

Theorem 3: Assume that $[A|B]$ is stabilizable. Then the NCS with SNR channel model can be stabilized by state feedback under channel resource allocation, if and only if $\mathfrak{C} > h(A)$.

Proof: We only need to show

$$\inf_{\gamma_1,\ldots,\gamma_m>0} \inf_{F \text{ stabilizing}} \mathfrak{C} = h(A).$$

In light of Remark 2 in the Appendix, we can simply assume that A is anti-stable. We first prove that for a given stabilizing state feedback gain F and a scaling matrix Γ, the total channel capacity $\mathfrak{C} \geq h(A)$. Denote $\tilde{B} = B\Gamma^{-1}\Sigma$ and $\tilde{F} = \Sigma^{-1}\Gamma F$, then $[A|\tilde{B}]$ is stabilizable and $\tilde{F}$ is a stabilizing state feedback gain for this system. Let $\tilde{T}(z) = \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}$. By Lemma 3, we have

$$\begin{aligned}&\frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^* d\omega\right)\\ =&\frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right) \geq h(A).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathfrak{C} &= \frac{1}{2}\log\prod_{i=1}^{m}\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}\\ &\geq \frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right)\\ &\geq h(A),\end{aligned}$$

where the first inequality follows from Hadamard's inequality [26]: for any $m\times m$ positive definite matrix $Q = [q_{ij}]$, it holds $\det(Q) \leq \Pi_{i=1}^m q_{ii}$ and the equality holds if and only if Q is diagonal.

Without loss of generality, $[A|B]$ is assumed to have the Wonham decomposition given by (4), where each subsystem $[A_i|b_i]$ is stabilizable with state dimension n_i. Now we show that for any $\epsilon > 0$, if the total capacity constraint is given by $h(A) + \epsilon$, then one can find an allocation of this constraint among the input channels in the form

$$\{h(A_1) + \frac{\epsilon}{m}, \ldots, h(A_m) + \frac{\epsilon}{m}\} \tag{21}$$

and simultaneously design a feedback gain F such that the closed-loop system is stable and each channel capacity satisfies the constraint $\mathfrak{C}_i < h(A_i) + \frac{\epsilon}{m}$. The allocation of channel capacities is done indirectly by choosing an appropriate scaling matrix Γ. Specifically, let

$$\Gamma^{-1}\Sigma = \mathrm{diag}\{1, \eta, \ldots, \eta^{m-1}\}$$

with η a small positive real number. Define

$$P = \mathrm{diag}\{I_{n_1}, \eta I_{n_2}, \ldots, \eta^{m-1}I_{n_m}\}.$$

Then

$$\begin{aligned}\tilde{T}(z) &= \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}\\ &= \tilde{F}P(zI - P^{-1}AP - P^{-1}\tilde{B}\tilde{F}P)^{-1}P^{-1}\tilde{B},\end{aligned}$$

where

$$P^{-1}AP = \begin{bmatrix} A_1 & o(\eta) & \cdots & o(\eta)\\ 0 & A_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & o(\eta)\\ 0 & \cdots & 0 & A_m\end{bmatrix},$$

$$P^{-1}\tilde{B} = \begin{bmatrix} b_1 & o(\eta) & \cdots & o(\eta)\\ 0 & b_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & o(\eta)\\ 0 & \cdots & 0 & b_m\end{bmatrix}$$

and $\frac{o(\eta)}{\eta}$ approaches to a finite constant as $\eta \to 0$.

For any given total capacity constraint $h(A) + \epsilon$, we can always find an allocation of the total constraint in the form (21). By Corollary 1, for each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|T_i(z)\|_2^2 = M(A_i)^2 - 1$, where $T_i(z) = f_i(zI - A_i - b_i)^{-1}b_i$. Now let $F = \tilde{F}P = \mathrm{diag}\{f_1, f_2, \ldots, f_m\}$, then

$$\begin{aligned}\mathfrak{C}_i &= \frac{1}{2}\log\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}\\ &= \frac{1}{2}\log\left\{I+\frac{1}{2\pi}\int_0^{2\pi}\tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^* d\omega\right\}_{ii}\\ &= \frac{1}{2}\log\left(1 + \|T_i(z)\|_2^2\right) + o(\eta)\\ &= \frac{1}{2}\log M(A_i)^2 + o(\eta)\\ &= h(A_i) + o(\eta).\end{aligned}$$

By choosing a sufficiently small $\eta > 0$, the actual channel capacities can be made to satisfy the constraints $\mathfrak{C}_i < h(A_i) + \frac{\epsilon}{m}$ for $i = 1, \ldots, m$. Apparently, the total capacity satisfies $\mathfrak{C} < h(A) + \epsilon$. ■

In light of Remark 1, the above design of F in the proof is the same as that in the R-SER model case. We want to emphasize that the channel capacity allocation is done indirectly here by choosing the scaling matrix Γ, i.e., by adjusting the transmission power in the different input channels. The difference from the setup in [29], [42] lies in that the total channel capacity, rather than the total transmission power, is assumed to be constrained. Once again, we witness the benefits brought in by the channel-controller co-design. With the additional design freedom gained by the channel resource allocation, the problem of networked stabilization becomes well formulated and admits a nice analytic solution.

VII. An illustrative example

In this section, we provide a numerical example to illustrate how an NCS is stabilized by channel-controller co-design under each of the three channel models. We apply the SER model and R-SER model to the logarithmic quantizer and alternative logarithmic quantizer respectively. Because the associated uncertainties are static, the inequality (5) and (16) are necessary for quadratic stability, but may not be necessary for stability of the closed-loop system. Nevertheless, the example illustrates the use of channel-controller co-design, that is the main purpose of this section. For the sake of numerical computation, we take the logarithm with base 2 in the example.

Consider an unstable system $[A|B]$ with

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $[A|B]$ is stabilizable. However, $[A|\alpha_1 B_1 + \alpha_2 B_2]$ is not stabilizable for any $\alpha_1, \alpha_2 \in \mathbb{R}$, since the matrix $\begin{bmatrix} \lambda I - A & \alpha_1 B_1 + \alpha_2 B_2 \end{bmatrix}$ loses row rank when $\lambda = 2$. This fact implies that it is impossible to convert $[A|B]$ to a stabilizable single-input system by a linear combination of the two inputs. Note that $[A|B]$ is already in the Wonham decomposition form with

$$A = \mathrm{diag}\{A_1, A_2\}, \quad b_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}', \quad b_2 = 1,$$

where $A_1 = \mathrm{diag}\{4, 2\}$ and $A_2 = 2$. The topological entropy of the plant is

$$h(A) = h(A_1) + h(A_2) = \log_2(4 \times 2) + \log_2 2 = 3 + 1 = 4.$$

A. SER model

Let the total capacity be given by $\mathfrak{C} = 4 + 2 \times 10^{-2} > h(A)$. We allocate the capacity among the two input channels as $\mathfrak{C}_1 = 3 + 10^{-2} > h(A_1), \mathfrak{C}_2 = 1 + 10^{-2} > h(A_2)$. Then the two logarithmic quantizers are characterized by $\delta_1 = 2^{-\mathfrak{C}_1} = 0.124$ and $\delta_2 = 2^{-\mathfrak{C}_2} = 0.497$.

To design the state feedback gain, we solve the $\mathcal{H}_\infty$ optimal $T(z)$ for the following two single-input systems:

$$[A_1|b_1] = \left[\begin{array}{cc|c} 4 & 0 & 1 \\ 0 & 2 & 1 \end{array}\right] \text{ and } [A_2|b_2] = [2|1], \tag{22}$$

yielding the optimal feedback gains $f_1 = \begin{bmatrix} -6.667 & 1.333 \end{bmatrix}$ and $f_2 = -2$, respectively. Let

$$F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix} -6.667 & 1.333 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

With this co-design of input channels and state feedback gain F, the closed-loop evolution of the plant states starting from an initial condition stimulated by an impulse is shown in Fig. 10. The state converges to zero asymptotically.

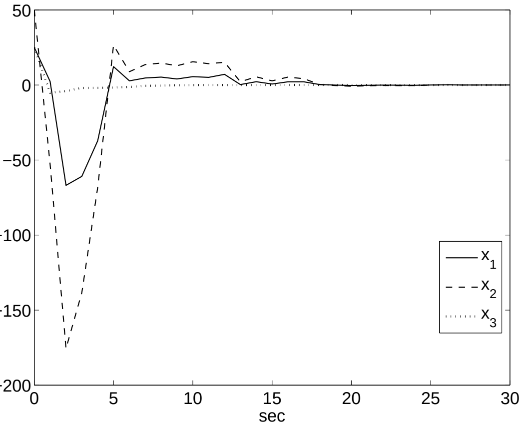

Fig. 10: State evolution with logarithmic quantizer.

B. R-SER model

Let the total capacity be given by $\mathfrak{C} = 4 + 2 \times 10^{-2}$. We allocate the capacity among the two input channels as $\mathfrak{C}_1 = 3 + 10^{-2}, \mathfrak{C}_2 = 1 + 10^{-2}$ that is the same allocation as in the SER model case. Then the two alternative logarithmic quantizers are again characterized by $\delta_1 = 2^{-\mathfrak{C}_1} = 0.124$ and $\delta_2 = 2^{-\mathfrak{C}_2} = 0.497$, although they have different physical meanings from those in the SER model case.

The design of state feedback gain is different from the SER model case. We solve the $\mathcal{H}_\infty$ optimal $S(z)$ instead of $\mathcal{H}_\infty$ optimal $T(z)$ for the two single-input systems in (22) and obtain the optimal feedback gains $f_1 = \begin{bmatrix} -6.563 & 1.313 \end{bmatrix}$ and $f_2 = -1.5$, respectively. Let

$$F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix} -6.563 & 1.313 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}.$$

With this co-design of input channels and state feedback gain F, the closed-loop evolution of the plant states starting from an initial condition stimulated by an impulse is shown in Fig. 11. The state converges to zero asymptotically.

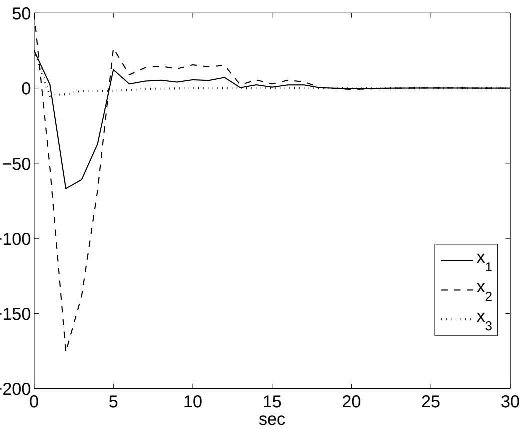

Fig. 11: State evolution with alternative logarithmic quantizer.

C. SNR model

As mentioned before, the channel resource allocation in this case is done by choosing the scaling matrix Γ, which is different from the previous two models. Specifically, let

$$\Gamma^{-1}\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}.$$

To design the state feedback gain, we solve the $\mathcal{H}_2$ optimal $T(z)$ for the two single-input systems in (22) and obtain the optimal gains $f_1 = \begin{bmatrix} -6.563 & 1.313 \end{bmatrix}$ and $f_2 = -1.5$, which are identical to those in the R-SER model case. Let

$$F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix} -6.563 & 1.313 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}.$$

Under this feedback controller, the numerical results on the channel capacities are summarized in Table I. Since η is a design parameter, its values are also listed.

TABLE I: Simulation results with SNR model.

η	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}$
10^{-1}	$3 + 1.6 \times 10^{-2}$	1	$4 + 1.6 \times 10^{-2}$
10^{-2}	$3 + 1.7 \times 10^{-4}$	1	$4 + 1.7 \times 10^{-4}$
10^{-3}	$3 + 1.7 \times 10^{-6}$	1	$4 + 1.7 \times 10^{-6}$

We can see that as $\eta \to 0$, the total capacity $\mathfrak{C} \to h(A)$. In other words, for any $\epsilon > 0$, when the total channel capacity constraint is given by $h(A) + \epsilon$, we can always simultaneously design the state feedback gain F and find an allocation of the capacities among the input channels to stabilize the closed-loop system. To demonstrate more clearly how the channel resource allocation is done, let the total capacity constraint be specifically given by $4+4\times10^{-4}$. Then we allocate this constraint among the two input channels as $\{3+2\times10^{-4}, 1+2\times10^{-4}\}$. Now we choose $\eta = 10^{-2}$ and use the state feedback gain F designed above. Under this channel-controller co-design, the channel capacities $\mathfrak{C}_1 = 3+1.7\times10^{-4} < 3+2\times10^{-4}$, $\mathfrak{C}_2 = 1 < 1+2\times10^{-4}$ as shown in Table I. The total capacity satisfies the constraint $\mathfrak{C} = 4+1.7\times10^{-4} < 4+4\times10^{-4}$. One realization of the stochastic closed-loop evolution of the plant states is shown in Fig. 12. Note that the closed-loop system states converge to a stationary stochastic process, not to zero, due to the stationary noises in the input channels.

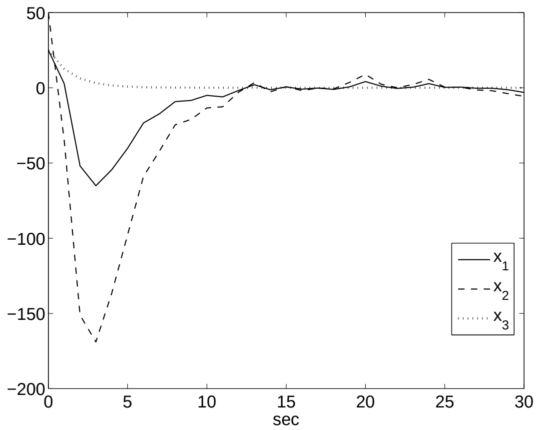

Fig. 12: State evolution with SNR channel model.

VIII. CONCLUSION

In this paper, we study the stabilization of networked multi-input systems with imperfect input channels, i.e., the controller-actuator channels. Each input channel is modeled in three different ways, i.e., the SER model, the R-SER model and the SNR model. One of the novelties of this paper is in the introduction of the R-SER model that is motivated from the alternative logarithmic quantizer in Fig. 5. The advantage of this alternative quantizer over the commonly used one in the literature is that it leads to the minimization of $\|S(z)\|_\infty$ instead of $\|T(z)\|_\infty$ as for the commonly used quantizer. The minimization of $\|S(z)\|_\infty$ shares a common optimal state feedback gain with the minimization of $\|S(z)\|_2$ and $\|T(z)\|_2$. Hence this optimization problem is intrinsically multi-objective. In contrast, the minimization of $\|T(z)\|_\infty$ conflicts that of $\|S(z)\|_\infty, \|T(z)\|_2, \|S(z)\|_2$ in the sense that its optimal solution may worsen the others which may potentially cause unexpected problems.

The main contribution of this paper is in the introduction of the channel resource allocation to find the least total channel capacity required for multi-input networked stabilization under each channel model. We assume that the total channel capacity is determined by the available resource which can be allocated among different input channels. With this additional design freedom, the controller designer should also participate in the channel design rather than passively take the channels given by the system designer. The overall process of channel resource allocation and the controller design constitutes channel-controller co-design. The main results in this paper can be summarized into a universal theorem: The state feedback stabilization can be accomplished by the channel-controller co-design, if and only if the total input channel capacity is greater than the topological entropy of the open-loop system. It has been justified in [37] that the topological entropy can serve as a measure of instability in a linear system. Finally, let us mention that the idea of channel resource allocation was first proposed in the conference paper [23]. Several other works [46], [8] have been carried out following this idea.

Only controller-actuator channels and state feedback control are considered in this paper. These serve as the starting point and are also of fundamental importance. The output feedback networked control which involves both sensor-controller channels and controller-actuator channels is much more challenging, and is currently under our study.

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REFERENCES

[1] P. Antsaklis and J. Baillieul (Guest Editors), "Special Issue on Networked Control Systems", *IEEE Trans. Automat. Contr.*, vol. 49, Sept. 2004.

[2] P. Antsaklis and J. Baillieul (Guest Editors), "Special Issue on Technology of Networked Control Systems", *Proc. of IEEE*, vol. 95, Jan. 2007.

[3] J. Baillieul, "Feedback coding for information based control: Operating near the data rate limit," in *Proc. 41st IEEE Conf. Decision Contr.*, pp. 3229-3236, Las Vegas, NV, 2002.

[4] J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, Operator Theory: Advances and Applications, vol. 45, Birkhäuser, Basel, Germany, 1990.

[5] T. Başar and P. Bernhard, $\mathcal{H}_\infty$ *Optimal Control and Related Minimax Design Problems: A Dynamic Gain Approach*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, MA, 1991.

[6] R. Bowen, "Entropy for group endomorphisms and homogeneous space", *Trans. of the American Mathematical Society*, vol 153, pp. 401-414, Jan. 1971.

[7] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Automat. Contr.*, vol. 52, pp. 1391-1403, Aug. 2007.

[8] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling", conditionally accepted by *Automatica*, 2012.

[9] W. Chen, L. Qiu, and G. Gu, "Stabilization of networked multi-input systems over AWGN channels with channel resource allocation", in *Proc. 9th IEEE Int. Conf. on Contr. & Automation*, pp. 680-685, Santiago, Chile, 2011.

[10] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, 1991.

[11] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, 2002.

[12] J. A. De Doná, G. C. Goodwin, and S. O. R. Moheimani, "Combining switching, over-saturation and scaling to optimise control performance in the presence of model uncertainty and input saturation", *Automatica*, vol. 38, pp. 1153-1162, 2002.

[13] N. Elia, "Remote stabilization over fading channels", *Systems & Control Letters*, vol. 54, pp. 237-249, 2005.

[14] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, Sept. 2001.

[15] J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky, "Stabilization with disturbance attenuation over a Gaussian channel", in *Proc. 46st IEEE Conf. Decision Contr.*, pp. 3958-3963, New Orleans, LA, 2007.

[16] J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky, "Minimum variance control over a Gaussian communication channel", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 1751-1765, Aug. 2011.

[17] J. S. Freudenberg, R. H. Middleton, and V. Solo, "Stabilization and disturbance attenuation over a Gaussian communication channel", *IEEE Trans. Automat. Contr.*, vol. 55, pp. 795-799, Mar. 2010.

[18] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Automat. Contr.*, vol. 50, pp. 1698-1711, Nov. 2005.

[19] H. Gao and T. Chen, "A new approach to quantized feedback control systems", *Automatica*, vol. 44, pp. 534-542, 2008.

[20] G. C. Goodwin, D. E. Quevedo, and E. I. Silva, "An introduction to networked control systems", *Proc. 6th Asian Control Conf.*, pp. 50-74, Bali, Indonesia, 2006.

[21] G. C. Goodwin, E. I. Silva, and D. E. Quevedo, "Analysis and design of networked control systems using the additive noise model methodology", *Asian Journal of Control*, vol. 12, pp. 443-459, July. 2010.

[22] M. Green and D. J. L. Limebeer, *Linear Robust Control*, Prentice-Hall, 1995.

[23] G. Gu and L. Qiu, "Networked stabilization of multi-input systems with channel resource allocation", *Reprints of the 17th IFAC World Congress*, Seoul, Korea, 2008.

[24] G. Gu and L. Qiu, "Robust stabilization of multiplicative/relative uncertain systems and networked feedback control", *Proc. 7th IEEE Int. Conf. on Contr. & Automation*, pp. 69-74, Christchurch, New Zealand, 2009.

[25] H. Haimovich and M. M. Seron, "On infimum quantization density for multiple-input systems", in *Proc. 44st IEEE Conf. Decision Contr.*, pp. 7692-7697, Seville, Spain, 2005.

[26] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[27] C. Y. Kao and S. R. Venkatesh, "Stabilization of linear systems with limited information-multiple input case", in *Proc. 2002 Amer. Control Conf.*, pp. 2406-2411, Anchorage, AK, 2002.

[28] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, New York, 1972.

[29] Y. Li, E. Tuncel, J. Chen, and W. Su, "Optimal tracking performance of discrete-time systems over an additive white noise channel", in *Proc. 48st IEEE Conf. Decision Contr.*, pp. 2070-2075, Shanghai, P.R.China, 2009.

[30] K. Mahler, "An application of Jensen's formula to polynomials", *Mathematika*, vol. 7, pp. 98-100, Dec. 1960.

[31] L. Moreau, "Stability of multiagent systems with time-dependent communication links", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 169-182, Feb. 2005.

[32] G. N. Nair and R. J. Evans, "Exponential stabilizability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585-593, 2003.

[33] G. N. Nair, F. Fagnani, S. Sampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview", *Proc. of IEEE*, vol. 95, pp. 108-137, Jan. 2007.

[34] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays", *Automatica*, vol. 34, pp. 57-64, 1998.

[35] P. Ogren, E. Fiorelli, and N. E. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1292-1302, Aug. 2004.

[36] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*, 2nd edition, Prentice Hall, 1997.

[37] L. Qiu, "Quantify the unstable", in *Proc. 19th International Symposium on Mathematical Theory of Networks and Systems*, pp. 981-986, Budapest, Hungary, 2010.

[38] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link-part I: scalar systems", *IEEE Trans. Inf. Theory*, vol. 52, pp. 3369-3395, Aug. 2006.

[39] A. V. Savkin, "Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control", *Automatica*, vol. 42, pp. 51-62, 2006.

[40] P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems", *Proc. 2001 Amer. Contr. Conf.*, vol. 2, pp. 1491-1496, Arlington, VA, 2001.

[41] J. S. Shamma, "Robust stability with time-varying structured uncertainty", *IEEE Trans. Automat. Contr.*, vol. 39, pp. 714-724, Apr. 1994.

[42] Z. Shu and R. H. Middleton, "Stabilization over power-constrained parallel Gaussian channels", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 1718-1724, July. 2011.

[43] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1453-1464, Sept. 2004.

[44] K. Tsumura, H. Ishii, and H. Hoshina, "Tradeoffs between quantization and packet loss in networked control of linear systems", *Automatica*, vol. 45, pp. 2963-2970, 2009.

[45] W. M. Wonham, "On pole assignment in multi-input controllable linear systems", *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 660-665, Dec. 1967.

[46] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels", *IEEE Trans. Automat. Contr.*, to appear, 2012.

[47] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems", *IEEE Control Syst. Mag.*, vol. 21, pp. 84-99, Feb. 2001.

[48] K. Zhou, with J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice-Hall, 1996.

Appendix: Proofs of Lemmas 2 and 3

A. *Proof of Lemma 2*

We first consider the case when A has no eigenvalues on the unit circle. Let λ be an unstable eigenvalue of A, then $\|S(\infty)\|_\infty = 1$ and $\|T(\lambda)\|_\infty = 1$. This shows that

$$\inf_{F \text{ stabilizing}} \|S(z)\|_\infty \geq 1 \text{ and } \inf_{F \text{ stabilizing}} \|T(z)\|_\infty \geq 1.$$

Taking $\theta = 1$ and $W = \gamma^{-1} I$ with $\gamma > 1$ in Lemma 1, we see that

$$\inf_{F \text{ stabilizing}} \|S(z)\|_\infty < \gamma$$

if and only if there exists a stabilizing solution $X \geq 0$ to the ARE

$$A'X(I + BB'X)^{-1}A = X \tag{23}$$

such that $I+B'XB<\gamma^2 I$. Taking $\theta=0$ and $W=\gamma^{-1}I$ with $\gamma>1$ in Lemma 1, we see that

$$\inf_{F \text{ stabilizing}} \|T(z)\|_\infty < \gamma$$

if and only if there exists a stabilizing solution $\tilde{X} \geq 0$ to the ARE

$$A'\tilde{X}\left[I + B(1-\gamma^{-2})B'\tilde{X}\right]^{-1} A = \tilde{X} \tag{24}$$

such that $B'\tilde{X}B < \gamma^2 I$. Notice that the stabilizing solution to an ARE, if exists, is unique. Comparing (23) with (24), it is easy to see that $X = (1-\gamma^{-2})\tilde{X}$. Then $X \geq 0$ if and only if $\tilde{X} \geq 0$. Also $I + B'XB < \gamma^2 I$ if and only if $B'\tilde{X}B < \gamma^2 I$. Therefore, the above two necessary and sufficient conditions for the cases when $\theta = 1$ and $\theta = 0$ are actually the same. In other words, the existence of the desired X and $\tilde{X}$ are equivalent, implying that

$$\inf_{F \text{ stabilizing}} \|S(z)\|_\infty = \inf_{F \text{ stabilizing}} \|T(z)\|_\infty.$$

Consequently, it suffices to study the optimal sensitivity in the rest of the proof. In view of the requirement $I+B'XB < \gamma^2 I$, we have

$$\begin{aligned}\inf_{F \text{ stabilizing}} \|S(z)\|_\infty &= \inf_{F \text{ stabilizing}} \{\gamma : \|S(z)\gamma^{-1}\|_\infty < 1\} \\ &= \sqrt{\rho(I+B'XB)}.\end{aligned}$$

Without loss of generality, we can assume that

$$[A|B] = \left[\begin{array}{cc|c} A_\mathrm{s} & 0 & B_\mathrm{s} \\ 0 & A_\mathrm{u} & B_\mathrm{u}\end{array}\right], \tag{25}$$

where A_s is stable and A_u is anti-stable. By the existence and uniqueness of the stabilizing solution to ARE (23), we have

$$X = \begin{bmatrix} 0 & 0 \\ 0 & X_\mathrm{u}\end{bmatrix}$$

with X_u being the unique stabilizing solution to the ARE

$$\begin{aligned} X_\mathrm{u} &= A_\mathrm{u}'X_\mathrm{u}(I+B_\mathrm{u}B_\mathrm{u}'X_\mathrm{u})^{-1}A_\mathrm{u} \qquad (26)\\ &= A_\mathrm{u}'X_\mathrm{u}A_\mathrm{u} - A_\mathrm{u}'X_\mathrm{u}B_\mathrm{u}(I+B_\mathrm{u}'X_\mathrm{u}B_\mathrm{u})^{-1}B_\mathrm{u}'X_\mathrm{u}A_\mathrm{u}\end{aligned}$$

satisfying $I+B_\mathrm{u}'X_\mathrm{u}B_\mathrm{u} < \gamma^2 I$. Moreover, $X_\mathrm{u} > 0$ and has a closed form expression

$$X_\mathrm{u} = \left(\sum_{k=1}^{\infty} A_\mathrm{u}^{-k}B_\mathrm{u}B_\mathrm{u}'A_\mathrm{u}'^{-k}\right)^{-1}.$$

When $m = 1$, we have

$$\begin{aligned}\inf_{F \text{ stabilizing}} \|S(z)\|_\infty &= \sqrt{1+B'XB} = \sqrt{\det(I+BB'X)} \\ &= \sqrt{\det(I+B_\mathrm{u}B_\mathrm{u}'X_\mathrm{u})} \\ &= \sqrt{\det(A_\mathrm{u})\det(A_\mathrm{u}')} = M(A).\end{aligned}$$

When $m > 1$, the eigenvalues of $I + B'XB$ are all real and no less than one. Thus

$$\begin{aligned}\inf_{F \text{ stabilizing}} \|S(z)\|_\infty &= \sqrt{\rho(I+B'XB)} \leq \sqrt{\det(I+B'XB)} \\ &= \sqrt{\det(I+BB'X)} = M(A).\end{aligned}$$

To prove the lower bound, let $\Phi(z) = \gamma^{-1}S(z^{-1})$, then $\Phi(z)$ is analytic inside the unit circle and $\|\Phi(z)\|_\infty < 1$. Let λ_1 be an eigenvalue of A with magnitude $\rho(A)$. Then

$$\begin{aligned}\det S(\lambda_1) &= \det(I + F(\lambda_1 I - A - BF)^{-1}B) \\ &= \det\left(I + (\lambda_1 I - A - BF)^{-1}BF\right) \\ &= \det\left((\lambda_1 I - A - BF)^{-1}(\lambda_1 I - A)\right) = 0.\end{aligned}$$

Therefore, the matrix $\Phi(\lambda_1^{-1}) = \gamma^{-1}S(\lambda_1)$ has eigenvalue at 0. In other words, there exists $x \in \mathbb{C}^m$ with $x^*x = 1$ such that

$$\Phi(\lambda_1^{-1})x = 0. \tag{27}$$

Moreover, since $S(\infty) = I$, it follows that

$$\Phi(0)I = \gamma^{-1}I. \tag{28}$$

By the theory of tangential Nevanlinna-Pick interpolation [4], such a matrix function $\Phi(z)$ satisfying (27) and (28) exists if and only if the extended Pick matrix

$$\begin{bmatrix} \dfrac{x^*x - 0}{1-\lambda_1^{-1}\lambda_1^{*-1}} & \dfrac{x^* - 0}{1-0} \\ \dfrac{x-0}{1-0} & \dfrac{I-\gamma^{-2}I}{1-0}\end{bmatrix} = \begin{bmatrix} \dfrac{1}{1-\rho(A)^{-2}} & x^* \\ x & (1-\gamma^{-2})I\end{bmatrix} > 0.$$

Since $(1-\gamma^{-2})I > 0$, the above inequality holds if and only if

$$\frac{1}{1-\rho(A)^{-2}} - \frac{1}{1-\gamma^{-2}} > 0.$$

This implies that $\gamma > \rho(A)$ and completes the proof for the lower bound.

Finally, we address the case when A has eigenvalues on the unit circle. In this case, neither (23) nor (24) has a stabilizing solution. Nevertheless, we can let $A_\epsilon = (1+\epsilon)A$ with $\epsilon > 0$ such that A_ϵ has the same number of eigenvalues inside the unit circle as A but no eigenvalues on the unit circle. By applying the procedure above to system $[A_\epsilon|B]$ and taking the limit $\epsilon \to 0$, we can obtain the same result as in the case when A does not have eigenvalues on the unit circle.

B. *Proof of Lemma 3*

We first consider the following set

$$\Omega = \left\{\frac{1}{2\pi}\int_0^{2\pi}(e^{j\omega}I - A - BF)^{*^{-1}}F^*F(e^{j\omega}I-A-BF)^{-1}d\omega : A+BF \text{ is stable}\right\},$$

which is a subset of the partially ordered set of $n \times n$ positive semi-definite matrices. The infimum of Ω, denoted as $\inf\Omega$, is the greatest lower bound of Ω. The least element of Ω, if exists, is an element of Ω which is less than or equal to any other element of Ω. Apparently, Ω contains a least element if and only if $\inf\Omega \in \Omega$. Denote the closure of Ω by $\overline{\Omega}$.

We will show that when $[A|B]$ is stabilizable, $\inf \Omega \in \overline{\Omega}$. Firstly, consider the case when A has no eigenvalues on the unit circle. By Parseval's identity [36], we have

$$\frac{1}{2\pi}\int_0^{2\pi}(e^{j\omega}I - A - BF)^{*^{-1}}F^*F(e^{j\omega}I - A - BF)^{-1}d\omega$$
$$= \sum_{k=0}^{\infty}(A + BF)^{*k}F^*F(A + BF)^k.$$

The right-hand side of the above equation is precisely the solution to

$$P = (A + BF)'P(A + BF) + F'F$$

that is a discrete-time Lyapunov equation. This fact implies that

$$\Omega = \{P : P = (A + BF)'P(A + BF) + F'F, \quad A + BF \text{ is stable}\}.$$

It is well known from the $\mathcal{H}_2$ optimal control theory [28] that $\inf \Omega = X$, where X is the unique stabilizing solution to ARE (23), and the corresponding optimal gain F is given by

$$F = -B'X(I + BB'X)^{-1}A. \tag{29}$$

Moreover, we have $\inf \Omega = X \in \Omega$, which implies that X is in fact the least element of Ω .

If A has eigenvalues on the unit circle, the desired feedback gain (29) cannot be achieved. Therefore, the least element of Ω does not exist in this case. As in the proof of Lemma 2, we let $A_\epsilon = (1 + \epsilon)A$ with $\epsilon > 0$ such that A_ϵ has the same number of eigenvalues inside the unit circle as A but no eigenvalues on the unit circle. We also define the subset Ω_ϵ correspondingly. Applying the above derivation to system $[A_\epsilon|B]$ yields that Ω_ϵ has a least element given by the stabilizing solution X_ϵ to ARE

$$A_\epsilon' X_\epsilon(I + BB'X_\epsilon)^{-1}A_\epsilon = X_\epsilon.$$

Taking the limit $\epsilon \to 0$, we get $\lim_{\epsilon\to 0} X_\epsilon = X$, where X is the unique semi-stabilizing solution to (23) in the sense that all the eigenvalues of $A - BB'X(I + BB'X)^{-1}A$ lie in the closed unit disk. This implies that $\inf \Omega \in \overline{\Omega}$ holds.

Now we prove the hold of equality (2) and inequality (3). Without loss of generality, we only need to consider the case when A is anti-stable. For the equality (2), using $\inf \Omega = X$, where $X > 0$ is the stabilizing solution to (23), together with the fact that the log determinant function is operator monotone increasing on the cone of positive definite matrices yields

$$\inf_{F \text{ stabilizing}} \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^*T(e^{j\omega})d\omega\right)$$
$$= \frac{1}{2}\log\det(I + B'XB).$$

Since

$$\det(I + B'XB) = \det(I + BB'X) = \det(X^{-1}A'XA)$$
$$= \det(A')\det(A) = M(A)^2,$$

it follows that

$$\inf_{F \text{ stabilizing}} \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^*T(e^{j\omega})d\omega\right)$$
$$= \frac{1}{2}\log M(A)^2 = h(A)$$

which concludes the proof of the equality (2).

We proceed to prove the inequality (3). For an arbitrary F such that $A + BF$ is stable, the matrix $A' + F'B'$ is also stable. This implies that the system $[A'|F']$ is stabilizable. Moreover, B' is a stabilizing state feedback gain. In this case, the complementary sensitivity function of system $[A'|F']$ is $T'(z) = B'(zI - A' - F'B')^{-1}F'$. In view of (2), we have

$$\frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T'(e^{j\omega})^*T'(e^{j\omega})d\omega\right)$$
$$= \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{-j\omega})T(e^{-j\omega})^*d\omega\right)$$
$$= \frac{1}{2}\log\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})T(e^{j\omega})^*d\omega\right)$$
$$\geq h(A).$$

Since the choice of stabilizing F is arbitrary, the inequality (3) follows.

Remark 2: The proofs of Lemma 2 and Lemma 3 show that the eigenvalues of A on the unit circle have no effect on the infimum of $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norm of $S(z)$ and $T(z)$. In addition, the system $[A|B]$ can be assumed to have decomposition (25). By decomposing F into $\begin{bmatrix} F_s & F_u \end{bmatrix}$ with compatible dimensions, $F_s = 0$ can be used in minimizing both the $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ norm of $S(z)$ and $T(z)$. As a result, the stable eigenvalues of A also have no effect on the optimization value. Therefore, we can simply assume that A is anti-stable without loss of generality when we encounter optimization of $S(z)$ and $T(z)$.

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