Staircase Pattern Constrained Zero-One Matrix Completion with Uncertainties and Its Applications

Extended Abstract

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Abstract— **In this paper, we analyze the existence conditions of a special class of constrained zero-one matrices with uncertainties. Classically, it asks how to characterize a class of zero-one matrices with given row sum and column sum vectors. The solution is given by a concise majorization inequality, widely known as the Gale-Ryser theorem. The authors further explore it in a more complicated case. Firstly, some fixed zeros or ones are given in a prescribed staircase pattern of positions. Secondly, each row or column sum is not exactly given but described by a boundary interval. It shows that such a constrained zero-one matrix exists if and only if two structure tensors are jointly nonnegative, which is consistent with the previous majorization condition.**

I. INTRODUCTION

The majorization theory is a young but rather powerful mathematical tool in a wide range of applications. To verify this explicitly, a number of names will be listed for their pioneering works regarding this theory. See, for instance, Lorenz [1], Dalton [2], Hardy, Littlewood, Pólya [3], Marshall, Olkin, and Arnold [4], etc. Nowadays, a relatively thorough treatise is the book – "Inequalities: Theory of Majorization and Its Applications" by Marshall, Olkin, and Arnold [4]. Apart from a summary of basic results, it elaborates some related mathematical applications of the majorization order, like combinatorial analysis, geometric inequalities, stochastic ordering, and matrix factorization, etc. To some extent, it also mentions the potential applications in many related areas, such as engineering, economics, medicines, and politics.

As we can see, majorization has arisen in diverse relevant topics. Among them, we are particularly interested in those of a basically combinatorial nature. For this end, we delve into a zero-one matrix completion problem. Given two nonnegative integer vectors with the same vector sum, a classic question then arises – whether there exists a zero-one matrix such that one of the two vectors refers to the column sums, while the other corresponds to the row sums of the matrix? For a numerical way to check the existence of such zero-one matrices, readers can refer to the Ryser's algorithm [5]. In 1957, Ryser [6] and Gale [7] respectively derived a concise inequality in terms of the majorization order as the analytic necessary and sufficient condition in order that a zero-one matrix exists with given row and column sum vectors. To be specific, the existence is established if and only if one of the two given vectors is majorized by the conjugate of the other vector. Note that the conjugate herein refers to the conjugate partition other than the complex conjugate of a complex number. See [6]–[10] for more details. Motivated by this celebrating problem, a number of pioneering works have been conducted along this research line.

Fulkerson considered a variant by further requiring the zero trace of a zero-one matrix in [11]. A more complicated case is investigated by Anstee in [12], where it characterizes a class of matrices with prescribed row and column sums and covering a specific matrix whose column sums are at most one. Brualdi and Dahl studied the existence of a zero-one matrix with given line sums and a fixed zero block located at one of the corners of the matrix in [13], where the existence was characterized in terms of a "structure matrix". In [14], a q-MFP (Matrix Feasibility Problem) was proposed where there were identical fixed zero blocks on the main diagonal. The authors of this paper further explored the line-sum constrained zero-one matrix completion with fixed zeros forming a staircase pattern in [15], where the existence is characterized by the nonnegativity of a "structure tensor".

In this paper, we further extend the results along the lines we have introduced. In most practical cases, the column/row sums are not exactly given but described by upper and lower bounds. Is there a zero-one matrix whose column and row sums are within the prescribed bounds? We generalize the results in [15] to deal with the staircase pattern constrained zero-one matrix completion with uncertainties. It shows that such a constrained zero-one matrix exists if and only two corresponding structure tensors are jointly nonnegative. This result is consistent with the previous result, studied in [16]. Specifically, when there are no fixed zeros (or ones), this condition is reduced to that two associated majorization inequalities jointly hold. Some by-products are also presented to give hints to construct such a required zero-one matrix.

In addition to a purely mathematical problem, the constrained zero-one matrix completion problems can find ap-

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$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

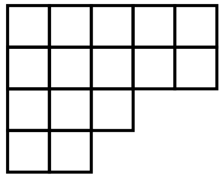

Fig. 1. A pattern matrix and its corresponding Young diagram.

plications in many areas such as electoral systems [14], smart grid [17], real-time systems [18], and discrete tomography [19]. In this paper, two applications will be mentioned to show the importance of the problems and results herein discussed. One is the supply-demand balance problem, while the other is the bidimensional position allocation problem.

II. PRELIMINARIES

In this section, we will introduce some preliminary knowledge. Let $\mathbb{R}$ (*resp.* $\mathbb{N}$) denote real numbers (*resp.* nonnegative integers). Denote the transpose of a matrix F by F'. We reserve E as the matrix of a proper size with all the elements equal to one. For a vector $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}' \in \mathbb{R}^n$, its nonincreasing rearrangement is $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$.

Definition 1: For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we write $\boldsymbol{x} \prec^w \boldsymbol{y}$ and say that $\boldsymbol{x}$ is weakly supermajorized by $\boldsymbol{y}$ if

$$\sum\nolimits_{i=s}^{n} x_{[i]} \geq \sum\nolimits_{i=s}^{n} y_{[i]}, \ \forall s = 1, 2, \ldots, n.$$

If further $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, then we write $\boldsymbol{x} \prec \boldsymbol{y}$ and say that $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$.

For $\boldsymbol{x} \in \mathbb{N}^n$, the conjugate of $\boldsymbol{x}$ is denoted by $\boldsymbol{x}^*$, which should be distinguished from the complex conjugate of a complex number. The jth element x_j^* of $\boldsymbol{x}^*$ is given by the number of elements of $\boldsymbol{x}$ that are no less than j. Note that the size of the conjugate is adjustable, when necessary.

A Young diagram is a collection of boxes, arranged in left-justified rows with nonincreasing row sizes. A Young diagram is said to be of shape $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ if its row sizes are given by λ_i, $i = 1, 2, \ldots, m$. Obviously, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Moreover, the row size and the column size vectors of a Young diagram are the conjugate of each other. For example, a Young diagram of shape $\lambda = \{5 \quad 5 \quad 3 \quad 2\}$ is shown in the right of Fig. 1. See that $\begin{bmatrix} 5 & 5 & 3 & 2 \end{bmatrix}'$ and $\begin{bmatrix} 4 & 4 & 3 & 2 & 2 \end{bmatrix}'$ are the conjugate of each other.

III. PROBLEM FORMULATION AND RESULTS

Let $\boldsymbol{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_m \end{bmatrix}'$ and $\boldsymbol{h} = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}'$ be two nonnegative integer vectors. The pattern matrix F is a zero-one matrix of size $m \times n$, signifying the positions of fixed zeros in a required zero-one matrix. Denote the class of $m \times n$ zero-one matrices with the row sum vector $\boldsymbol{r}$, the column sum vector $\boldsymbol{h}$, and the pattern matrix F by $\mathscr{A}(\boldsymbol{r}, \boldsymbol{h}, F)$, which refers to matrices satisfying the following constraints:

$$\sum_{j=1}^{n} A(i,j) = r_i, \ \forall i = 1, 2, \ldots, m;$$

$$\sum_{i=1}^{m} A(i,j) = h_j, \ \forall j = 1, 2, \ldots, n;$$

$$A(i,j) \in \{0,1\}, \ \forall i = 1, 2, \ldots, m \ \& \ j = 1, 2, \ldots, n;$$

$$0 \leq A(i,j) \leq F(i,j), \ \forall i = 1, 2, \ldots, m \ \& \ j = 1, 2, \ldots, n.$$

We call $\{\boldsymbol{r}, \boldsymbol{h}, F\}$ the structural information of the desired zero-one matrices. If we disregard the pattern matrix F, the famous Gale-Ryser theorem tells us that $\mathscr{A}(\boldsymbol{r}, \boldsymbol{h}, E) \neq \emptyset$ if and only if $\boldsymbol{h} \prec \boldsymbol{r}^*$. In our setup, we require that the pattern matrix F presents a staircase pattern, e.g., the pattern matrix in the left of Fig. 1. In the previous paper [15], the authors used the language of Young diagrams to describe the positions of fixed zeros. Obviously, there is a one-to-one correspondence between a staircase pattern matrix F and a Young diagram of the shape as the row sum vector of F, as illustrated in Fig. 1. In this regard, we hereinafter identify the pattern matrix F with the shape of the corresponding Young diagram. For instance, we can identify the staircase pattern matrix F in Fig. 1 with the shape $\{5 \quad 5 \quad 3 \quad 2\}$. Now, we can define a structure tensor $W(\boldsymbol{r}, \boldsymbol{h}, F)$ by way of the results in [15]. Write $W(\boldsymbol{r}, \boldsymbol{h}, F) \geq 0$ if every element of the tensor is nonnegative. The explicit expression of $W(\boldsymbol{r}, \boldsymbol{h}, F)$ is out of the scope of this paper, and readers can easily derive it by following the procedures in [15]. Without loss of generality, we assume that the order of the structure tensor is τ. Then, we rewrite the main theorem in [15] as the following lemma.

Lemma 3.1: The matrix class $\mathscr{A}(\boldsymbol{r}, \boldsymbol{h}, F) \neq \emptyset$ if and only if $W(\boldsymbol{r}, \boldsymbol{h}, F) \geq 0$ and $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} h_j$.

Instead of giving the exact line sums, we herein give the four boundary vectors of row and column sums respectively:

$$\underline{\boldsymbol{r}} = \begin{bmatrix} \underline{r}_1 & \underline{r}_2 & \cdots & \underline{r}_m \end{bmatrix}',$$

$$\overline{\boldsymbol{r}} = \begin{bmatrix} \overline{r}_1 & \overline{r}_2 & \cdots & \overline{r}_m \end{bmatrix}',$$

$$\underline{\boldsymbol{h}} = \begin{bmatrix} \underline{h}_1 & \underline{h}_2 & \cdots & \underline{h}_n \end{bmatrix}',$$

$$\overline{\boldsymbol{h}} = \begin{bmatrix} \overline{h}_1 & \overline{h}_2 & \cdots & \overline{h}_n \end{bmatrix}'.$$

Define $\mathscr{A}(\underline{\boldsymbol{r}}, \overline{\boldsymbol{r}}, \underline{\boldsymbol{h}}, \overline{\boldsymbol{h}}, F)$ as the class of matrices satisfying the following constraints:

$$\underline{r}_i \leq \sum_{j=1}^{n} A(i,j) \leq \overline{r}_i, \ \forall i = 1, 2, \ldots, m; \quad (1)$$

$$\underline{h}_j \leq \sum_{i=1}^{m} A(i,j) \leq \overline{h}_j, \ \forall j = 1, 2, \ldots, n; \quad (2)$$

$$A(i,j) \in \{0,1\}, \ \forall i = 1, 2, \ldots, m \ \& \ j = 1, 2, \ldots, n; \quad (3)$$

$$0 \leq A(i,j) \leq F(i,j), \ \forall i = 1, 2, \ldots, m \ \& \ j = 1, 2, \ldots, n, \quad (4)$$

Mathematically, we herein investigate the staircase pattern constrained zero-one matrix completion with uncertainties by way of finding out the necessary and sufficient conditions, under which $\mathscr{A}(\underline{\boldsymbol{r}}, \overline{\boldsymbol{r}}, \underline{\boldsymbol{h}}, \overline{\boldsymbol{h}}, F)$ is nonempty.

With the help of Lemma 3.1, we state the main theorem of this paper in the following.

Theorem 3.2: The matrix class $\mathscr{A}(\underline{\boldsymbol{r}}, \overline{\boldsymbol{r}}, \underline{\boldsymbol{h}}, \overline{\boldsymbol{h}}, F) \neq \emptyset$ if and only if $W(\underline{\boldsymbol{r}}, \overline{\boldsymbol{h}}, F) \geq 0$ and $W(\underline{\boldsymbol{h}}, \overline{\boldsymbol{r}}, F') \geq 0$ jointly hold.

In what follows, we will discuss a simple analogue of the above theorem for illustration. Given two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, we claim that $[\underline{a}, \overline{a}] \cap [\underline{b}, \overline{b}] \neq \emptyset$ if and only if $\overline{a} \geq \underline{b}$ and $\overline{b} \geq \underline{a}$ jointly hold. The first inequality $W(\underline{\boldsymbol{r}}, \overline{\boldsymbol{h}}, F) \geq 0$ means that there exists a zero-one matrix, whose row sum vector is $\underline{r}$ and column sum vector is no more than $\overline{h}$. That is analogous to $\overline{a} \geq \underline{b}$. Correspondingly, $W(\underline{\boldsymbol{h}}, \overline{\boldsymbol{r}}, F') \geq 0$ is analogous to $\overline{b} \geq \underline{a}$. As we can see, if every element of the pattern matrix F

is one, then both structure tensors in Theorem 3.2 have order one. By a little algebra, we obtain the following corollary from Theorem 3.1.

Corollary 1: The class $\mathscr{A}(\underline{\boldsymbol{r}},\overline{\boldsymbol{r}},\underline{\boldsymbol{h}},\overline{\boldsymbol{h}},E) \neq \emptyset$ if and only if $\overline{\boldsymbol{h}} \prec^{w} \underline{\boldsymbol{r}}^{*}$ and $\overline{\boldsymbol{r}} \prec^{w} \underline{\boldsymbol{h}}^{*}$ jointly hold.

This result in the above corollary was first shown in [16]. When the pattern matrix is reduced to E, the necessary and sufficient condition is reduced from the nonnegativity of two structure tensors to two inequalities in the weak majorization order. Thus, in a mathematical perspective, our result can be regarded as an extension of the majorization theory.

Before we proceed, it is necessary to define two supplementary vector classes as follows:

$$\{\boldsymbol{\Delta r}\} = \Big\{ \Delta \boldsymbol{r} \in \mathbb{N}^{m} \mid W(\underline{\boldsymbol{h}}, \underline{\boldsymbol{r}} + \Delta \boldsymbol{r}, F') \geq 0 \ \& \ \sum_{i=1}^{m} \Delta r_i = \Big| \min_{k_1,k_2,\dots,k_\tau} W_{k_1 k_2 \cdots k_\tau}(\underline{\boldsymbol{h}}, \underline{\boldsymbol{r}}, F') \Big| \Big\},$$

$$\{\boldsymbol{\Delta h}\} = \Big\{ \Delta \boldsymbol{h} \in \mathbb{N}^{n} \mid W(\underline{\boldsymbol{r}}, \underline{\boldsymbol{h}} + \Delta \boldsymbol{h}, F) \geq 0 \ \& \ \sum_{j=1}^{n} \Delta h_j = \Big| \min_{k_1,k_2,\dots,k_\tau} W_{k_1 k_2 \cdots k_\tau}(\underline{\boldsymbol{r}}, \underline{\boldsymbol{h}}, F) \Big| \Big\}.$$

The following corollary follows directly from a constructive proof of Theorem 3.2.

Corollary 2: For every $\Delta \boldsymbol{r} \in \{\boldsymbol{\Delta r}\}$ and every $\Delta \boldsymbol{h} \in \{\boldsymbol{\Delta h}\}$, we have $\mathscr{A}(\underline{\boldsymbol{r}} + \Delta \boldsymbol{r}, \underline{\boldsymbol{h}} + \Delta \boldsymbol{h}, F) \neq \emptyset$.

It follows naturally that $\sum_{i=1}^{m}(\Delta r_i + \underline{r}_i) = \sum_{j=1}^{n}(\Delta h_j + \underline{h}_j)$. In addition, the above corollary implies an instructive way to construct some matrices in $\mathscr{A}(\underline{\boldsymbol{r}},\overline{\boldsymbol{r}},\underline{\boldsymbol{h}},\overline{\boldsymbol{h}},F)$ with the minimum number of ones. Firstly, apply modified flow algorithms to get supplementary vectors $\Delta \boldsymbol{r}$ and $\Delta \boldsymbol{h}$, respectively. Then, obtain a zero-one matrix with the row sum and column sum vectors as $\underline{\boldsymbol{r}} + \Delta \boldsymbol{r}$ and $\underline{\boldsymbol{h}} + \Delta \boldsymbol{h}$, respectively. This can be done by some maximal flow algorithms and/or various Ryser-like algorithms as shown in [5]. Based on this subclass of the class $\mathscr{A}(\underline{\boldsymbol{r}},\overline{\boldsymbol{r}},\underline{\boldsymbol{h}},\overline{\boldsymbol{h}},F)$, more matrices satisfying the constraints (1)-(4) can be found by proper searching algorithms. In view of Corollary 2, it is not necessary to address the issues regarding both the row side and the column side simultaneously when constructing a required zero-one matrix. Specifically, we can fix the column sum vector as the lower boundary vector firstly and then enlarge the row sum vector such that the corresponding structure tensor is elementwise nonnegative. Similarly, we again fix the row sum vector as the lower boundary vector firstly and then enlarge the column sum vector such that the corresponding structure tensor is nonnegative. It happens that the lower boundary vectors together with their associated supplementary vectors match with each other well.

IV. APPLICATIONS

The first application is concerned with a supply-demand balance problem in smart grid. In [20], the duration-deadline jointly differentiated energy services were put forward. The original problem is merely related to matrix classes like $\mathscr{A}(\boldsymbol{r},\boldsymbol{h},F)$, which is elaborated in the following. In this power market, the power is delivered over n consecutive time slots. The available power in each time slot j corresponds to the jth column sum. In each time slot, load i either obtains the power delivered at a fixed rate or gets no charged. Without loss of generality, we assume that the fixed power delivery rate is one. Moreover, the load can never be charged after its required deadline. In the paper [20], the supply profile is exactly given while the duration requirements are also exactly given. This corresponds to the special case that the upper and lower bounds for the column/row sums coincide. However, in some practical cases, the supply profile (the column sum vector) is described by a upper bound and a lower bound together. In addition, the duration requirement of each load (the row sum) is given by an interval as well. The modified problem requires an exploration of matrix classes like $\mathscr{A}(\underline{\boldsymbol{r}},\overline{\boldsymbol{r}},\underline{\boldsymbol{h}},\overline{\boldsymbol{h}},F)$. In view of this, the result in this paper allows a more practical investigation for the duration-deadline jointly differentiated energy services.

The second application is in relation to the bidimensional position allocation problem. Each column is associated with a specific job while each row corresponds to a group of candidates. The first group of candidates are qualified for all the jobs, while the ith group of candidates are qualified for all the jobs that the $(i+1)$th group are qualified for. This allows the pattern matrix to present a staircase. Moreover, a job cannot be assigned to two or more candidates from the same group. In this application, generally, the number of candidates of each group is exactly given, i.e., $\underline{\boldsymbol{r}} = \overline{\boldsymbol{r}}$, but the available positions of each job is described by two boundary vectors $\underline{\boldsymbol{h}}$ and $\overline{\boldsymbol{h}}$. As we can see, Theorem 3.2 can help us check whether there exists a feasible position allocation.

V. CONCLUSIONS

In this paper, we explore the staircase pattern constrained zero-one matrix completion with uncertainties. A necessary and sufficient condition is found out to check whether some special zero-one matrix exists or not. In particular, it has two main requirements. On one hand, the fixed positions of the required zero-one matrix present a staircase pattern. On the other hand, the row and column sums of the required matrix should be within prescribed boundary intervals. We apply the nonnegativity of two associated structure tensors to characterize the class of matrices satisfying the prescribed requirements. If we omit the pattern matrix requirement, the necessary and sufficient condition can be reduced to two inequalities in terms of the weak majorization order. Furthermore, to clarify the significance of this work, we relate the staircase pattern constrained zero-one matrix completion with uncertainties to two promising applications, i.e., the supply-demand balance problem in the smart grid and the bidimensional position allocation problem. In the future, the authors want to relax the requirement of the pattern matrix from a staircase zero-one matrix to a general zero-one matrix.

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