A Dynamical Network Framework with Application to Stability of Power Networks

Dan Wang, Wei Chen, Sei Zhen Khong, Ji Liu, Tamer Başar, and Li Qiu

***Abstract*— We advocate and motivate in this paper a general framework for studying dynamical networks with both node (agent) and edge (communication) dynamics. Specifically, the agents interact with each other via transfer incident systems and dynamical communication channels. The potential of using the proposed framework to model network problems and architectures is demonstrated by a concrete example. The framework facilitates the study of stability and performance of various network dynamics. An application to angle stability of electrical power networks involving locally positive feedbacks is discussed as an illustrating example.**

I. Introduction

Networks are ubiquitous in daily life. Examples include social networks [1], biological networks [2], data networks [3], power networks [4], and transportation networks [5]. The analysis and manipulation of such networks has started during the last century and attracted researchers from various fields. During the past decades, efforts have been devoted to studying dynamical networks, including opinion dynamics in social networks [6], structure controllability and observability of complex networks [7], and stability analysis in electrical power networks [8].

In earlier studies of dynamical networks, the nodes are modeled as dynamical systems while the edges are often modeled as static weights. Such a simplification to static weights brings in considerable technical convenience. However, considering dynamical nodes but static edges appears to be a practice that foregoes half of the dynamical modelling capabilities at our disposal. It is rather desirable and reasonable to assign equal dynamical importance to the nodes and edges, and study networks with both node and edge dynamics. As a matter of fact, nodes and edges have a duality relationship and their roles can even be interchanged in some contexts [9]. Furthermore, numerous problems commonly examined in the literature naturally involve networks with edge dynamics. The signal flow graph [10] is one such example, which is a tool widely used in electronics engineering. Another example is communication networks, where signals are transmitted through communication channels that are inherently dynamical.

More research works on dynamical networks with both node and edge dynamics have emerged recently, signifying their growing importance. For instance, Nepusz and Vicsek [11] studied the controllability of edge dynamics by considering that of node dynamics on line graphs. Pates [12] derived robust, decentralised, and scalable stability criteria for networks of heterogeneous systems. Khong et al. [13], [14] investigated robust synchronisation of heterogeneous agents with dynamical interconnections. Bürger et al. [15] introduced a cooperative control framework involving refined passive systems for network analysis and optimal design.

This paper is driven by the motivation that edge dynamics are of essential importance in the study of dynamical networks. Building upon [15], we establish a general framework for studying dynamical networks with both node and edge dynamics. Specifically, the agents interact with each other via transfer incident systems and dynamical communication channels. The potential of using the proposed framework to model network problems and architectures is demonstrated by a concrete example. The framework facilitates the study of stability and performance of various network dynamics. An application to angle stability of electrical power networks involving locally positive feedbacks is discussed.

Notation: We denote by A' the transpose of a matrix A. We use $\mathbf{1}$ to denote the vector with all entries equal to 1, while the size of the vector is to be understood from the context. The symbol diag denotes the diagonal operation.

The rest of the paper is organized as follows. Section II introduces a general dynamical network framework with both node and edge dynamics. Section III provides some preliminary knowledge. One specific example of networks that fit within our framework is illustrated in Section IV. One specific engineering application is studied in Section V. Some concluding remarks follow in Section VI.

II. Problem Formulation

A general dynamical network is depicted in Fig. 1. In this framework,

$$P = \mathrm{diag}\{P_1, P_2, \dots, P_n\},$$
$$W = \mathrm{diag}\{W_1, W_2, \dots, W_m\},$$

where P_i is a scalar transfer function representing the dynamics of agent i, and W_k is a scalar transfer function

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representing the dynamics of edge e_k. The concatenated output of all the agents $y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \end{bmatrix}'$ is transmitted to the edges through the transfer incident system T, which determines the information available to each edge. Analogously, the output of the dynamical edges $z(t) = \begin{bmatrix} z_1(t) & z_2(t) & \dots & z_m(t) \end{bmatrix}'$ provides feedback information to the agents. The transfer incident system R determines how the outputs of edges are aggregated and fed back to the agents. The structures of the transfer incident pairs T and R reflect the network topology. Together with W, they form the network dynamics K.

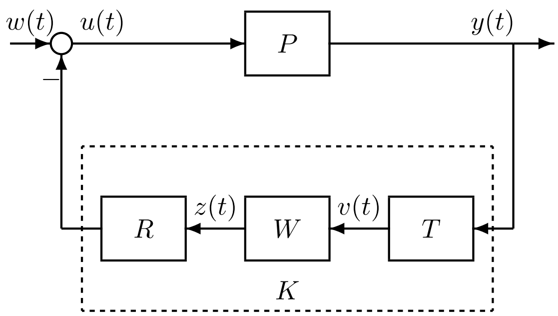

Fig. 1. Block diagram of a dynamical network

While this framework is closely related to that in [15], it comes with some major differences. In particular, locally positive feedback is allowed in the edge dynamics here. Moreover, we formulate T (R, respectively) as a dynamical system and it may represent an encoder (a decoder, respectively) in a communication network. In [15], T and R specialize to the incidence matrix of an undirected graph.

Note that this framework can be generalized to the case when P is a full transfer matrix, meaning that the agents are physically coupled. Nevertheless, we consider only decoupled agents in this paper. In Section IV, the formulation of one specific network that fits within this framework will be presented. Some preliminary knowledge is first provided in the next section.

III. Preliminaries

A. Graph theory

A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$ and a set of edges $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$. A dynamically weighted graph is a graph in which each edge $e_k = (i, j) \in \mathcal{E}$ is associated with a dynamical weight W_k, where W_k is a scalar transfer function. A static graph can be viewed as a special case, where W_k is a real-valued number. A graph is undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$.

Let $W = \mathrm{diag}\{W_1, W_2, \dots, W_m\}$ denote the dynamical weight matrix. Node i (j, respectively) is the head node (tail node, respectively) of the edge e_k if $e_k = (i, j)$. The incidence matrix $E \in \mathbb{R}^{n \times m}$ is defined as:

$$[E]_{ik} = \begin{cases} 1, & \text{if } i \text{ is the head node of } e_k, \\ -1, & \text{if } i \text{ is the tail node of } e_k, \\ 0, & \text{otherwise.} \end{cases}$$

If a graph is undirected, we can assign an arbitrary direction to each edge and get the incidence matrix similarly. An important property of the incidence matrix is $E'\mathbf{1} = 0$. Replacing all the 1's (-1's, respectively) in E by 0's yields the head incidence matrix H (tail incidence matrix F, respectively).

Now, with the dynamical weight matrix W, the incidence matrix E, and the head incidence matrix H defined above, the dynamical Laplacian L for directed graphs can be defined as

$$L = HWE'$$

and that for undirected graphs it can be defined as

$$L = EWE'.$$

Note that L is a transfer function matrix and $L(j\omega)$ has a zero eigenvalue corresponding to an eigenvector $\mathbf{1}$ for all $\omega \geq 0$. In the case when all the edges have constant positive weights, L reduces to the conventional Laplacian matrix, to which a substantial literature has been dedicated [16].

B. Passivity

Consider a linear time-invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$, and $y(t) \in \mathbb{R}^p$. The corresponding transfer function representation of the system (1) is given by

$$G(s) = C(sI - A)^{-1}B + D. \tag{2}$$

Definition 1 ([17]): An $n \times n$ proper rational transfer function matrix $G(s)$ is called positive real if

(i) poles of all elements of $G(s)$ are in $\mathrm{Re}(s) \leq 0$,
(ii) for all $\omega \geq 0$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^*(j\omega)$ is positive semidefinite, and
(iii) any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is Hermitian and positive semidefinite.

The transfer function $G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$.

With the definition of positive realness as above, the passive systems can be defined as follows:

Definition 2 ([18]): An LTI system of the form (1) is passive (strictly passive, respectively) if and only if its transfer function $G(s)$ given by (2) is positive real (strictly positive real, respectively).

We also need the notion of passive matrices as below.

Definition 3: A matrix $A \in \mathbb{R}^{n\times n}$ is said to be passive (strictly passive, respectively) if $x'Ax \geq 0$ ($x'Ax > 0$, respectively) for all $x \in \mathbb{R}^n$ and $x \neq 0$.

Note that a symmetric passive (strictly passive, respectively) matrix A is also called positive semidefinite (positive definite, respectively).

Passive systems enjoy many useful properties. We list two of them below for further use. For more details, the readers may refer to [19], [20].

Lemma 1 ([19, Theorem 2.33]): Consider two systems G_1 and G_2 in a negative feedback configuration, as shown in Fig. 2. The closed-loop system is passive if G_1 and G_2 are passive.

Lemma 2 ([19, Proposition 2.47]): Consider two systems G_1 and G_2 in a negative feedback configuration, as shown in Fig. 2. The closed-loop system is asymptotically stable if G_1 is passive and G_2 is strictly passive.

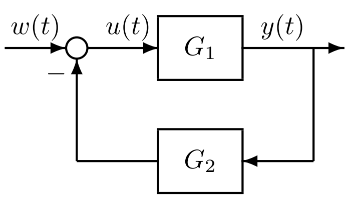

Fig. 2. Feedback configuration

IV. An Example

A signal flow graph, also called Mason graph [10], is a network of directed branches which connect at certain nodes. The nodes represent system variables, and the branches represent functional connections between pairs of nodes. See Fig. 3 for a simple example. A signal flow graph can be interpreted as a signal transmission system in which each node stands for a tiny station. The station receives signals via the incoming branches, aggregates the information in some manner, and then transmits the result along each outgoing branch.

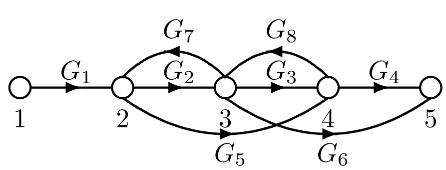

Fig. 3. A simple signal flow graph

A signal flow graph can be formulated within the framework proposed in Section II. Consider a signal flow graph with n nodes and m branches. Each branch is associated with a scalar transfer function G_k describing its dynamics. As shown in Fig. 4, for such a signal flow graph, the input-output relation of node i is simply a unity transfer function, i.e.,

$$y_i(s) = P_i u_i(s) = u_i(s).$$

The dynamics of edge e_k takes the following form:

$$z_k(s) = G_k v_k(s).$$

The transfer incident systems reduce to the graph tail incidence matrix and transpose of the head incidence matrix, i.e.,

$$R = F \text{ and } T = H',$$

implying that each edge takes the measurement of its head node as the input and injects the output to its tail node.

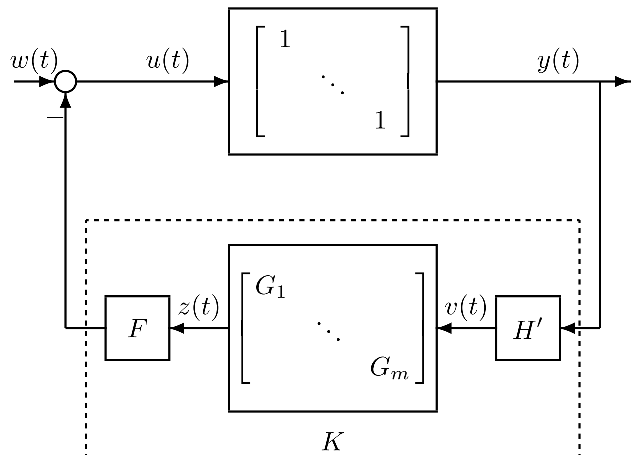

Fig. 4. Block diagram of signal flow graphs

V. An Application

In the previous sections, we have built a general framework for studying dynamical networks and shown a specific example that can be fitted into the framework. Certain questions then arise naturally: What can we further do with this framework? What problems can we formulate and study? What advantages does this framework provide in network analysis and design?

We envision that this framework will greatly facilitate the study of stability and performance of various network dynamics. As a case in point, we examine, in this section, an application to angle stability of power systems.

Small disturbance angle stability is a fundamental issue in electrical power systems. Here we study it within the proposed framework. For detailed discussions of this problem, we refer the readers to [8], [21], [22].

Consider a power network whose topology is described by an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$. Each node corresponds to a bus and each edge a transmission line. Denote by $Y_{ij} = Y_{ji}$ the admittance of the transmission line $(i, j) \in \mathcal{E}$. Denote the voltage magnitude and phase angle of bus i by V_i and θ_i, respectively.

The dynamics of phase angle θ_i at bus i is described as

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = p_i - \sum_{(i,j)\in\mathcal{E}} V_iV_jY_{ij}\sin(\theta_i - \theta_j), \ i \in \mathcal{V}. \quad (3)$$

In traditional synchronous machine based power networks, $m_i > 0$ and $d_i > 0$ are, respectively, the moment of inertia and the damping constant of the ith synchronous generator. When $m_i = 0$, the dynamics correspond to the inverter-based droop-controlled generators in a microgrid, where $d_i > 0$ is the reciprocal of frequency droop gain of the ith generator. In both cases, p_i is the nominal active power generation minus the electrical load at bus i. For technical simplicity, we consider the case when the voltage magnitude at each bus is a constant (not necessarily homogenous).

Denote by $(\theta^0, 0)$ an equilibrium point of system (3). We wish to examine the small-disturbance stability of the equilibrium point $(\theta^0, 0)$. To this end, we linearize the system (3) around $(\theta^0, 0)$ and get

$$m_i \Delta\ddot{\theta}_i + d_i \Delta\dot{\theta}_i = p_i - \sum_{(i,j)\in\mathcal{E}} V_i V_j Y_{ij} \cos(\theta_i^0 - \theta_j^0)(\Delta\theta_i - \Delta\theta_j).$$

This linearized power network dynamics can be put into our dynamical network framework, where the node dynamics are

$$P_i = \frac{1}{m_i s^2 + d_i s},$$

and the edge dynamics W_k are static weights

$$W_k = V_i V_j Y_{ij} \cos(\theta_i^0 - \theta_j^0).$$

Since the power network is undirected, the transfer incident systems R and T reduce to the incidence matrix E of the network, i.e., $T = E$ and $R = E$. Consequently, the whole network dynamics K is simply the Laplacian $L = EWE'$ of the network, as shown in Fig. 5.

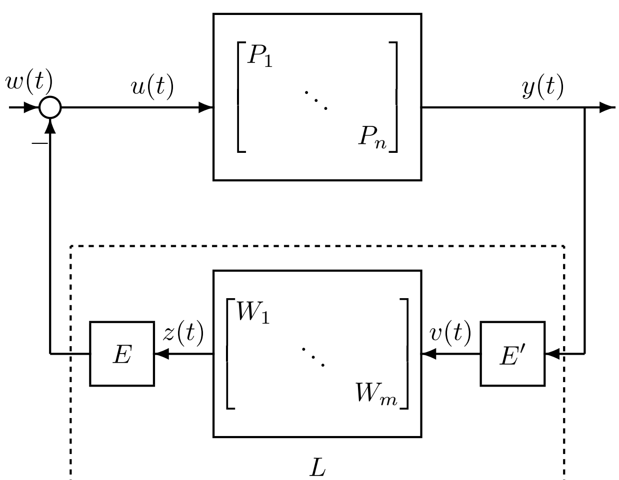

Fig. 5. Block diagram for studying the angle stability of a power system

Note that if for some $e_k = (i, j) \in \mathcal{E}$, the angle difference between θ_i^0 and θ_j^0 is larger than $\pi/2$, W_k will be negative, which corresponds to a positive feedback. Nevertheless, the following theorem shows that an equilibrium point can be small-disturbance stable even when some positive feedbacks exist, provided that L is positive semidefinite and has a simple zero eigenvalue.

Theorem 1: An equilibrium point $(\theta^0, 0)$ of (3) is small-disturbance stable if L is positive semidefinite and has a simple zero eigenvalue.

Proof: Consider a first-order system

$$d_i \dot{\theta}_i = p_i - \sum_{(i,j)\in\mathcal{E}} V_i V_j Y_{ij} \sin(\theta_i - \theta_j), \ i \in \mathcal{V}, \quad (4)$$

i.e., let m_i be zero in the system (3). It has been shown in [23, Theorem 5-1] that $(\theta^0, 0)$ is an equilibrium point of the system (3) if, and only if, θ^0 is an equilibrium point of the system (4). Furthermore, $(\theta^0, 0)$ of the system (3) is small disturbance stable if, and only if, θ^0 of the system (4) is small disturbance stable. Therefore, it suffices to analyze the small disturbance stability of θ^0 of the system (4). To this end, we linearize the system (4) around θ^0 and get

$$d_i \Delta\dot{\theta}_i = p_i - \sum_{(i,j)\in\mathcal{E}} V_i V_j Y_{ij} \cos(\theta_i^0 - \theta_j^0)(\Delta\theta_i - \Delta\theta_j).$$

These dynamics can be put into our framework. Then, the node dynamics are given by

$$\hat{P}_i = \frac{1}{d_i s},$$

which are passive. The edge dynamics are still static weights

$$W_k = V_i V_j Y_{ij} \cos(\theta_i^0 - \theta_j^0),$$

and the network dynamics are still the Laplacian matrix $L = EWE'$ of the network.

Let $Q \in \mathbb{R}^{(n-1)\times n}$ be a matrix whose rows form an orthonormal basis for $\mathrm{span}\{\mathbf{1}\}^\perp$. Define $\bar{P} = Q\hat{P}Q'$ and $\bar{L} = QLQ'$. Then, $\bar{L}$ is strictly passive if, and only if, L is passive with a simple zero eigenvalue. It has been widely recognized that the equilibrium point θ^0 of the system (4) is small disturbance stable if, and only if, the negative feedback connection of $\bar{P}$ and $\bar{L}$ is stable [8]. Then, the results follow from Lemma 2.

■

VI. Conclusions

In this paper, we have established a general framework for studying dynamical networks with both node and edge dynamics. The potential of using the proposed framework to model network problems and architectures has been demonstrated by a concrete example. The framework facilitates the study of stability and performance of various network dynamics. An application to angle stability of electrical power networks involving locally positive feedbacks has also been discussed.

In fact, the consensus problem in multi-agent systems can be formulated within our framework. Using the framework, a sufficient condition can be derived under which a group of heterogenous agents can reach consensus when they interact via dynamical communication channels. One can find the discussions on consensus problem in the longer version of the paper available from the authors.

We regard the framework considered in this paper as a starting point for studying the stability and performance of dynamical networks. We intend to explore more applications of this framework and investigate other network properties using this framework in the future.

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