Stability of General Dynamical Networks via Majorization of Phases

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***Abstract*— In this paper, we study the stability of a linear time-invariant dynamical network with both node and edge dynamics. It is demonstrated that at each frequency, the phases of the eigenvalues of the loop-gain transfer function are bounded from above (below, respectively) by the sum of the maximal (minimal, respectively) phases of the node and edge dynamics. Based on this property and the generalized Nyquist stability criterion, a sufficient condition for closed-loop stability is derived in terms of the phases of node and edge dynamics. This serves as an interesting starting point in developing a more general "small phase theorem".**

I. Background

Large-scale networks consisting of *dynamical* nodes and *static* edges have been the focus of numerous studies over the past decades. Take for instance opinion dynamics in social networks [1], structural controllability and observability of complex networks [2], and stability analysis in electrical power networks [3]. More recently, general dynamical networks containing both node and edge dynamics have attracted considerable attention [4]–[8] due to increasing awareness that edge dynamics are often of equal importance to node dynamics in the study of complex networks. Among others, one fundamental issue is to investigate the closed-loop stability of such general dynamical networks.

One of the most successful stories on closed-loop stability is the small gain theorem which can be found in many control textbooks [9]–[11]. It appears that much less attention has been paid to what is supposed to be a natural counterpart: a "small phase theorem". In the literature, there have been some interesting initiative attempts towards such a counterpart [12]–[14]. However, the development of a "small phase theorem" is far from settled to the extent that how to define the phases of a multivariable linear system remains obscure. This leads to a series of open questions: What is a suitable definition of phases of a matrix? What properties of the matrix phases shall we expect? How the phases are connected to the closed-loop stability?

The generalized Nyquist stability criterion [15] may provide a clue in understanding the connection between phases and closed-loop stability. It says that the closed-loop system is stable if and only if the eigenloci of the loop-gain transfer function do not encircle the point $(-1, 0)$. Hence, if we can show the phases of the eigenvalues of the loop-gain transfer function lie in the interval $(-\pi, \pi)$ for all frequencies, then the closed-loop system is stable.

Inspired by the questions raised above and the generalized Nyquist stability criterion, we study the closed-loop stability of dynamical networks by analysing the frequency response of the loop-gain transfer function of the network. We propose a definition of matrix phases for a class of matrices that are congruent to unitary matrices. It is demonstrated that at each frequency, a majorization relation exists between the phases of the eigenvalues of the loop-gain transfer function and the phases of the node and edge frequency responses. Then a sufficient condition for closed-loop stability is derived by using the generalized Nyquist stability criterion.

II. Problem Formulation

In this paper, we study the stability of a linear time-invariant dynamical network with both node and edge dynamics. We adopt the framework proposed in [8] and concentrate on the case of an undirected network topology as a starting point. In the network shown in Fig. 1,

$$P = \mathrm{diag}\{P_1, P_2, \dots, P_n\},$$
$$W = \mathrm{diag}\{W_1, W_2, \dots, W_m\},$$

where P_i is a proper stable scalar transfer function representing the dynamics of agent i, and W_k is a proper stable scalar transfer function representing the dynamics of edge e_k. The concatenated outputs of all the agents $y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \end{bmatrix}'$ is transmitted to the edges through a linear map represented by the matrix E', which is full rank and determines the information available to each edge. Analogously, the outputs of the dynamical edges $z(t) = \begin{bmatrix} z_1(t) & z_2(t) & \dots & z_m(t) \end{bmatrix}'$ are aggregated and fed back to the agents through the matrix E. The structure of the matrix E reflects the network topology. As a whole, the network dynamics are described by $K = EWE'$.

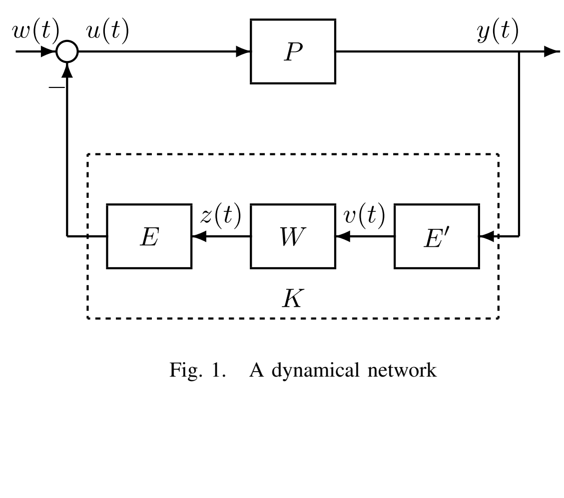

Fig. 1. A dynamical network

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In this paper, we denote by $\arg c \in (-\pi, \pi]$ the principal value of the argument of a complex number c. For $x \in \mathbb{C}^n$, we use $\arg x$ to denote $\begin{bmatrix}\arg x_1 & \arg x_2 & \dots & \arg x_n\end{bmatrix}'$. Given a matrix $A \in \mathbb{C}^{n\times n}$, the conjugate transpose of A is denoted by A^*, and the vector of eigenvalues of A is denoted by $\lambda(A) = \begin{bmatrix}\lambda_1(A) & \lambda_2(A) & \dots & \lambda_n(A)\end{bmatrix}'$.

III. Preliminaries

A. Majorization

The main result in this paper is based on a majorization relation between the phases of the eigenvalues of the loop-gain transfer function at each frequency and the phases of node and edge dynamics. Here, we briefly review some basic concepts in majorization theory. For an extensive treatment of majorization and its applications, one can refer to [16].

For $x, y \in \mathbb{R}^n$, we denote by $x^\uparrow$ and $y^\uparrow$ the rearranged versions of x and y so that their elements are arranged in a nondecreasing order. We say that x is majorized by y, denoted by $x \prec y$, if

$$\begin{aligned}\sum_{i=1}^{k} x_i^\uparrow &\geq \sum_{i=1}^{k} y_i^\uparrow, \quad \text{for } k = 1, \dots, n-1,\\ \sum_{i=1}^{n} x_i^\uparrow &= \sum_{i=1}^{n} y_i^\uparrow.\end{aligned}$$

Majorization defines a partial order on the evenness of the elements in two vectors when the averages of the elements are the same. If $x \prec y$, the elements of x are more even, or less spread out, than those of y.

B. Generalized Nyquist stability criterion

There have been several generalizations of the Nyquist stability criterion to deal with multi-input multi-output systems. One may refer to [15], [17]–[19]. Here we introduce the one proposed in [15], which will be used for the stability analysis of dynamical networks.

Lemma 1: Consider a proper stable transfer matrix $G(s) \in \mathbb{R}^{p\times p}$ with eigenloci[1] $(\lambda_i)_{i=1,\dots,p}$. The closed-loop system in Fig. 2 is stable if and only if the eigenloci $(\lambda_i)_{i=1,\dots,p}$ do not encircle the point $(-1, 0)$.

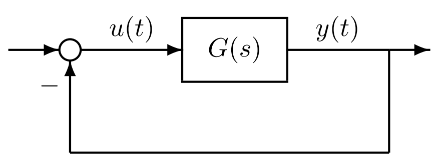

Fig. 2. A feedback system.

IV. Main Results

A. Phases of a matrix

A fundamental issue we are facing is to define the phases of a matrix. In the context of stability of dynamical network as shown in Fig. 1, a class of matrices that are congruent to unitary matrices are of particular interest. For such matrices, we propose the following definition of phases.

Definition 1: Let $A = T^*UT \in \mathbb{C}^{n\times n}$, where U is a unitary matrix and T is a nonsingular matrix. The phases of A, denoted by $\arg_i(A), i = 1, 2, \dots, n$, are defined to be the phases of the eigenvalues of U, i.e., $\arg_i(A) = \arg\ \lambda_i(U)$.

Hereinafter, we order the phases of matrix $A = T^*UT$ nondecreasingly, i.e., $\arg_1(A) \leq \arg_2(A) \leq \dots \leq \arg_n(A)$. Denote $\arg(A) = \begin{bmatrix}\arg_1(A) & \arg_2(A) & \dots & \arg_n(A)\end{bmatrix}'$. We also oder the eigenvalues of a general matrix $M \in \mathbb{C}^{n\times n}$ with the phases nondecreasing, i.e., $\arg\ \lambda_1(M) \leq \arg\ \lambda_2(M) \leq \dots \leq \arg\ \lambda_n(M)$.

This definition of matrix phases leads to the following nice property which plays a significant role in stability analysis of dynamical networks.

Theorem 1: Let $A = T^*UT \in \mathbb{C}^{n\times n}$ and $B = R^*VR \in \mathbb{C}^{n\times n}$, where T, R are nonsingular matrices and U, V are unitary matrices. If there exist $\theta_1, \theta_2 \in [-\pi, 0]$ such that

$$\begin{aligned}&\theta_1 < \arg_1(B) \leq \dots \leq \arg_n(B) < \theta_1 + \pi,\\ &\theta_2 < \arg_1(A) + \arg_1(B) \leq \dots\\ &\qquad\qquad \leq \arg_n(A) + \arg_n(B) < \theta_2 + \pi,\end{aligned}$$

then $\arg\ \lambda(AB) \prec \arg(A) + \arg(B)$.

The following lemma also plays an important role in network stability analysis.

Lemma 2 ([20]): Let $A = T^*UT \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{l\times l}$ be a principal submatrix of A. If there exists $\theta \in [-\pi, 0]$ such that $\theta < \arg_1(A) \leq \dots \leq \arg_n(A) < \theta + \pi$, then C is congruent to another unitary matrix V and

$$\arg_i(A) \leq \arg_i(C) \leq \arg_{n-l+i}(A) \text{ for } i = 1, 2, \dots, l.$$

Remark 1: In the literature, there are other attempts to define the phases of a matrix. For instance, the authors in [20] define the phases of a matrix A as the phases of the eigenvalues of the unitary part of A. That is, if $A = PU$ is the polar decomposition of A, where P is positive semidefinite and U is unitary, then $\arg_i(A) = \arg\ \lambda_i(U)$. This definition also possesses many properties that are analogous to the phase of a complex number and can be defined for any matrix. However, the majorization relation demonstrated in Theorem 1 does not hold.

B. Stability of dynamical networks

Applying Theorem 1 and Lemma 2 to the node dynamics P and edge dynamics W evaluated at some frequency $s = j\omega$, we have

$$\begin{aligned}\arg\ \lambda_1(P(j\omega)K(j\omega)) &\geq \arg_1(P(j\omega)) + \arg_1(W(j\omega)),\\ \arg\ \lambda_n(P(j\omega)K(j\omega)) &\leq \arg_n(P(j\omega)) + \arg_m(W(j\omega)).\end{aligned}$$

We can see from Lemma 1, if $\arg\ \lambda_i(P(j\omega)K(j\omega)) \in (-\pi, \pi)$ for all $\omega \in \mathbb{R}$ and $i = 1, \dots, n$, the eigenloci of $P(j\omega)K(j\omega)$ will never encircle $(-1, 0)$ point and thus the closed-loop system is stable. This can be achieved by restricting the sum of phases of the node and edge frequency responses to $(-\pi, \pi)$. Then we have the following theorem regarding the closed-loop stability of the network in Fig. 1.

[1]The eigenloci are obtained by rearrangements of the eigenvalue loci of $G(j\omega),\ \omega \in (-\infty, \infty)$.

Theorem 2: If the following inequalities

$$\begin{aligned}
&\arg_m(W(j\omega)) - \arg_1(W(j\omega)) < \pi, \\
&\arg_n(P(j\omega)) + \arg_m(W(j\omega)) \\
&\qquad\qquad - \arg_1(P(j\omega)) - \arg_1(W(j\omega)) < \pi
\end{aligned}$$

hold for all $\omega \in \mathbb{R}$, then the closed-loop system is stable.

This theorem characterizes a set of node dynamics and a set of edge dynamics in terms of the phases. For heterogenous node dynamics and edge dynamics within the respective sets, the closed-loop system is guaranteed to be stable even though the node or edge dynamics may have different orders.

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